Математички Билтен
47 (LXXIII) 1 (2023), 25–34

N-TUPLE ORBITS AND N-TUPLE WEAK ORBITS TENDING TO INFINITY

SONJA MANČEVSKA [1] AND MARIJA OROVČANEC [2]

Abstract. In this paper we give a sufficient condition for n pairwise commuting and bounded linear operators on an infinite dimensional complex Banach space X, which will imply that the space contains a dense set of vectors each with a corresponding n-tuple orbit tending to infinity. The same condition is sufficient to imply that the product of X and its dual space contains a dense set of pairs, each with a corresponding n-tuple weak orbit tending to infinity.

1. Introduction

Throughout this paper, unless otherwise stated, X will denote a complex, infinite dimensional Banach space, $B(X)$ the algebra of all bounded linear operators on X and X^* the dual space of X i.e., the space of all bounded linear functionals $x^* : X \to \mathbb{C}$. As usual, for $x \in X$ and $x^* \in X^*$ we will denote $\langle x, x^* \rangle := x^*(x)$. For the direct product $X \times X^*$ we assume that is a Banach space, in a sense of the direct sum of X and X^*, with one of the following norms: $\|(x,x^*)\|_\infty = \max\{\|x\|, \|x^*\|\}$ or $\|(x,x^*)\|_p = \left(\|x\|^p + \|x^*\|^p\right)^{1/p}$ for $1 \leq p < \infty$. $\mathbb{Z}_+$ will denote the set of all nonnegative integers and

$$\mathbb{Z}_+^n = \{(k_1, k_2, ..., k_n) : k_i \in \mathbb{Z}_+, 1 \leq i \leq n\}.$$

If $T_1, T_2, ..., T_n \in B(X)$ are pairwise commuting operators, the *n-tuple orbit* of the vector $x \in X$ (or the orbit of x under the n-tuple $\mathbf{T} = (T_1, T_2, ..., T_n)$) is the set

$$\mathrm{Orb}(\{T_i\}_{i=1}^n, x) = \mathrm{Orb}(\mathbf{T}, x) = \left\{ T_1^{k_1} T_2^{k_2} ... T_n^{k_n} x : (k_1, k_2, ..., k_n) \in \mathbb{Z}_+^n \right\}, \tag{1.1}$$

and the *n-tuple weak orbit* of the pair $(x, x^*) \in X \times X^*$ is the set

$$\begin{aligned} \mathrm{Orb}(\{T_i\}_{i=1}^n, x, x^*) &= \mathrm{Orb}(\mathbf{T}, x, x^*) \\ &= \left\{ \left\langle T_1^{k_1} T_2^{k_2} ... T_n^{k_n} x, x^* \right\rangle : (k_1, k_2, ..., k_n) \in \mathbb{Z}_+^n \right\}. \end{aligned} \tag{1.2}$$

By the definition given in [15], the n-tuple orbit (1.1) *tends to infinity* if

2010 *Mathematics Subject Classification.* Primary: 47A05 Secondary: 47A11, 47A25.
Key words and phrases. Banach spaces, bounded linear operators, n-tuple orbits, n-tuple weak orbits, spectral radius, approximate point spectrum, joint approximate point spectrum.

$$\lim_{k_i\to\infty}\left\|T_1^{k_1}T_2^{k_2}...T_n^{k_n}x\right\| = \infty, \text{ for every } k_j \in \mathbb{Z}_+,\ j \neq i, \text{ and every } 1 \leq i \leq n.$$

In [8] and [10] we gave a similar definition for n-tuple weak orbits: the n-tuple weak orbit (1.2) *tends to infinity* if

$$\lim_{k_i\to\infty}\left|\left\langle T_1^{k_1}T_2^{k_2}...T_n^{k_n}x, x^*\right\rangle\right| = \infty, \text{ for every } k_j \in \mathbb{Z}_+,\ j \neq i, \text{ and every } 1 \leq i \leq n.$$

For $n = 1$, the sets in (1.1) and (1.2) are sequences of form:

$$\mathrm{Orb}(T, x) = \{T^n x : n = 0, 1, 2, ...\} \subset X,$$

and

$$\mathrm{Orb}(T, x, x^*) = \{\langle T^n x, x^*\rangle : n = 0, 1, 2, ...\} \subset \mathbb{C}.$$

These sequences are usually referred as *single orbit* (or simply *orbit*) of the vector $x \in X$ and *single weak orbit* (or simply *weak orbit*) of the pair $(x, x^*) \in X \times X^*$ under the operator T, respectively. Clearly, if $\mathrm{Orb}(\{T_i\}_{i=1}^n, x)$ tends to infinity, then $\mathrm{Orb}(T_i, x)$ will also tend to infinity, for every $i \in \{1, 2, ..., n\}$. The same holds for the weak orbits: if $\mathrm{Orb}(\{T_i\}_{i=1}^n, x, x^*)$ tends to infinity, then $\mathrm{Orb}(T_i, x, x^*)$ will also tend to infinity, for every $i \in \{1, 2, ..., n\}$. As corollaries of the main results in [7]-[10], we've obtained that, if $T_1, T_2, ..., T_n \in B(X)$ are operators such that $r(T_i) > 1$, for all $i \in \{1, 2, ..., n\}$, then:

(i) X will contain a dense set D such that $\mathrm{Orb}(T_i, x)$ tends to infinity for all $x \in D$ and all $i \in \{1, 2, ..., n\}$ and if, in addition, the operators $T_1, T_2, ..., T_n$ are pairwise commuting and have at least one of the following properties:
   (P.1) T_i is bounded bellow, for every $i \in \{1, 2, ..., n\}$,
   (P.2) $(T_i^k - T_j^k)_{k\geq 0}$ is a norm bounded sequence, for all $i, j \in \{1, 2, ..., n\}$,
   then the m-tuple orbit $\mathrm{Orb}(\{T_{i_j}\}_{j=1}^m, x)$ will tend to infinity, for every $2 \leq m \leq n$, $1 \leq i_1 < i_2 < \ldots < i_m \leq n$ and $x \in D$,

(ii) $X \times X^*$ will contain a dense set D' such that $\mathrm{Orb}(T_i, x, x^*)$ tends to infinity, for all $(x, x^*) \in D'$ and all $i \in \{1, 2, ..., n\}$ and if, in addition, the operators $T_1, T_2, ..., T_n$ are pairwise commuting and have the property (P.2), then the m-tuple weak orbit $\mathrm{Orb}(\{T_{i_j}\}_{j=1}^m, x, x^*)$ will tend to infinity for every $2 \leq m \leq n$, $1 \leq i_1 < i_2 < \ldots < i_m \leq n$ and $x \in D'$.

The conditions (P.1) and (P.2) are quite rigorous. Moreover, for any operators $T_1, T_2, ..., T_n$ such that $r(T_i) > 1$, $i \in \{1, 2, ..., n\}$, the condition (P.2) will imply that all these operators must have the same spectral radius. In this paper we are going to show that vectors in X with n-tuple orbits and pairs in $X \times X^*$ with n-tuple weak orbits tending to infinity exist whenever $T_1, T_2, ..., T_n$ are pairwise commuting operators such that $r(T_i) > 1$ for every $i \in \{1, 2, ..., n\}$, without any additional conditions.

2. Preliminaries

As usual, for a single operator $T \in B(X)$, $\sigma(T)$, $\sigma_{\mathrm{p}}(T)$ and $\sigma_{\mathrm{ap}}(T)$ will denote the spectrum, the point spectrum and the approximate point spectrum of T.

If $\mathbf{T} = (T_1, T_2, ..., T_n)$ is an n-tuple of pairwise commuting operators on X, the joint approximate point spectrum (or the left approximate spectrum) of $\mathbf{T}$ is the set

$$\begin{aligned}\sigma_\pi(\mathbf{T}) &= \sigma_\pi(T_1, T_2, \dots, T_n) \\ &= \{(\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbb{C}^n : (\forall \varepsilon > 0)(\exists x \in X)\,\text{s.t. } \|x\| = 1 \wedge \\ &\qquad \|(T_i - \lambda_i)x\| < \varepsilon, 1 \le i \le n\}.\end{aligned}$$

For alternative equivalent definitions of the joint approximate point spectrum, we refer to [1], [3] and [11]. For every n-tuple of pairwise commuting operators $\mathbf{T} = (T_1, T_2, ..., T_n)$, $\sigma_\pi(\mathbf{T})$ is nonvoid and compact set ([3, Property 2]), which has the following property, usually referred as the spectral mapping theorem for the joint approximate point spectrum.

Theorem 1. [3, Theorem 1] *If $\mathbf{T} = (T_1, T_2, ..., T_n)$ is an n-tuple of pairwise commuting operators and f is an m-tuple of polynomials in n variables (so that $f(\mathbf{T})$ is defined and is an m-tuple of commuting operators), then $\sigma_\pi(f(\mathbf{T})) = f(\sigma_\pi(\mathbf{T}))$.*

Clearly, $\sigma_{\text{ap}}(T) = \sigma_\pi(T)$ for every operator $T \in B(X)$ and, by [4, Theorem 1],

$$r(T) = \max\left\{|\lambda| : \lambda \in \sigma_{\text{ap}}(T)\right\}, \text{ for every } T \in B(X). \tag{2.1}$$

We also need the following two results.

Theorem 2. [13, Theorem V.37.14] *Let X and Y be Banach spaces and $(T_n)_{n\ge1}$ be a sequence of operators in $B(X, Y)$. Let $(a_n)_{n\ge1}$ be sequence of positive numbers such that $\sum_{n=1}^{\infty} a_n < \infty$. Then there exists $x \in X$ such that $\|T_n x\| \ge a_n \|T_n\|$, for all $n \ge 1$. Moreover, it is possible to choose such an x in each ball in X of radius greater than $\sum_{n=1}^{\infty} a_n$.*

Theorem 3. [13, Theorem V.39.5] *Let X and Y be Banach spaces and $(T_n)_{n\ge1}$ be a sequence of operators in $B(X, Y)$. Let $(a_n)_{n\ge1}$ be sequence of positive numbers with $\sum_{n=1}^{\infty} a_n^{1/2} < \infty$. Then there are $x \in X$ and $y^* \in Y^*$ such that $|\langle T_n x, y^*\rangle| \ge a_n \|T_n\|$, for all $n \ge 1$. Moreover, given balls $B \subset X$ and $B^* \subset Y^*$ of radii greater than $\sum_{n\ge1} a_n^{1/2} < \infty$, then it is possible to find $x \in B$ and $y^* \in B^*$ with this property.*

3. N-TUPLE ORBITS TENDING TO INFINITY

Theorem 4. *If $\mathbf{T} = (T_1, T_2, \dots, T_n)$ is an n-tuple of pairwise commuting operators on an infinite dimensional complex Banach space X such that $r(T_i) > 1$, for every $1 \le i \le n$, then there is a dense set $D_1 \subset X$ such that the n-tuple orbit $\text{Orb}(\{T_i\}_{i=1}^n, x)$ tends to infinity for every $x \in D_1$.*

Proof. Let $x_0 \in X$ and $\varepsilon > 0$. Since $r(T_i) > 1$, for all $1 \le i \le n$, by (2.1) there are $\lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{C}$ such that $\lambda_i \in \sigma_{\text{ap}}(T_i)$ and $|\lambda_i| = r(T_i) > 1, 1 \le i \le n$. Let $q \in \mathbb{R}$ and $C > 0$ are such that

$$1 < q < \min\{|\lambda_1|, |\lambda_2|, ..., |\lambda_n|\}, \tag{3.1}$$

$$C\left(\frac{q}{q-1}\right)^n < \varepsilon. \tag{3.2}$$

If $p_1 < p_2 < \ldots < p_n$ are the first n prime numbers, let $g : \mathbb{Z}_+^n \to \mathbb{Z}_+$ be the injective mapping defined with $g(k_1, k_2, \ldots, k_n) = p_1^{k_1} p_2^{k_2} \ldots p_n^{k_n}$ and let

$$a_{g(k_1,k_2,\ldots,k_n)} = \frac{C}{q^{k_1+k_2+\ldots+k_n}} > 0, \text{ for } (k_1, k_2, \ldots, k_n) \in \mathbb{Z}_+^n,$$

$$S_{g(k_1,k_2,\ldots,k_n)} = T_1^{k_1} T_2^{k_2} \ldots T_n^{k_n}, \text{ for } (k_1, k_2, \ldots, k_n) \in \mathbb{Z}_+^n.$$

By the first inequality in (3.1) and by (3.2) we have

$$\begin{aligned} \sum_{(k_1,k_2,\ldots,k_n)\in\mathbb{Z}_+^n} a_{g(k_1,k_2,\ldots,k_n)} &= \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \ldots \sum_{k_n=0}^{\infty} \frac{C}{q^{k_1+k_2+\ldots+k_n}} \\ &= C \prod_{i=1}^{n} \left(\sum_{k_i=0}^{\infty} \frac{1}{q^{k_i}} \right) = C \left(\frac{q}{q-1} \right)^n < \varepsilon. \end{aligned}$$

Hence, applying Theorem 2 on the sequence $\{a_{g(k_1,k_2,\ldots,k_n)} : (k_1, k_2, \ldots, k_n) \in \mathbb{Z}_+^n\}$ and the sequence $\{S_{g(k_1,k_2,\ldots,k_n)} : (k_1, k_2, \ldots, k_n) \in \mathbb{Z}_+^n\}$, we can find a vector $x \in X$ such that $\|x - x_0\| < \varepsilon$ and

$$\begin{aligned} \left\| T_1^{k_1} T_2^{k_2} \ldots T_n^{k_n} x \right\| &\geq \frac{C}{q^{k_1+k_2+\ldots+k_n}} \left\| T_1^{k_1} T_2^{k_2} \ldots T_n^{k_n} \right\| \\ &\geq \frac{C}{q^{k_1+k_2+\ldots+k_n}} r(T_1^{k_1} T_2^{k_2} \ldots T_n^{k_n}), \forall (k_1, k_2, \ldots, k_n) \in \mathbb{Z}_+^n. \end{aligned} \tag{3.3}$$

If $(k_1, k_2, \ldots, k_n) \in \mathbb{Z}_+^n$ and $p_{k_1,k_2,\ldots,k_n} : \mathbb{C}^n \to \mathbb{C}$ is the polynomial defined with,

$$p_{k_1,k_2,\ldots,k_n}(z_1, z_2, \ldots, z_n) = z_1^{k_1} z_2^{k_2} \ldots z_n^{k_n},$$

then, by Theorem 1,

$$\begin{aligned} \sigma_{\text{ap}}(T_1^{k_1} T_2^{k_2} \ldots T_n^{k_n}) &= \sigma_{\text{ap}}(p_{k_1,k_2,\ldots,k_n}(T_1, T_2, \ldots, T_n)) \\ &= p_{k_1,k_2,\ldots,k_n}(\sigma_\pi(T_1, T_2, \ldots, T_n)) \\ &= \left\{ z_1^{k_1} z_2^{k_2} \ldots z_n^{k_n} : (z_1, z_2, \ldots, z_n) \in \sigma_\pi(T_1, T_2, \ldots, T_n) \right\}. \end{aligned} \tag{3.4}$$

On the other hand, if $p_i : \mathbb{C}^n \to \mathbb{C}$ are the polynomials defined with,

$$p_i(z_1, z_2, \ldots, z_n) = z_i, \quad 1 \leq i \leq n,$$

then (again by Theorem 1),

$$p_i(\sigma_\pi(T_1, T_2, \ldots, T_n)) = \sigma_\pi(p_i(T_1, T_2, \ldots, T_n)) = \sigma_{\text{ap}}(T_i), \text{ for all } 1 \leq i \leq n. \tag{3.5}$$

Since $\lambda_i \in \sigma_{\text{ap}}(T_i)$, (3.5) implies that there are $\mu_1^{(i)}, \ldots, \mu_{i-1}^{(i)}, \mu_{i+1}^{(i)}, \ldots, \mu_n^{(i)} \in \mathbb{C}$ such that,

$$(\mu_1^{(i)}, \ldots, \mu_{i-1}^{(i)}, \lambda_i, \mu_{i+1}^{(i)}, \ldots, \mu_n^{(i)}) \in \sigma_\pi(T_1, T_2, \ldots, T_n).$$

Then, by (2.1), (3.3) and (3.4),

$$\begin{aligned}
&\left\| T_1^{k_1} T_2^{k_2} ... T_n^{k_n} x \right\| \\
&\geq \frac{C}{q^{k_1+k_2+...+k_n}} r(T_1^{k_1} T_2^{k_2} ... T_n^{k_n}) \\
&= \frac{C}{q^{k_1+k_2+...+k_n}} \max \left\{ |\lambda| : \lambda \in \sigma_{\text{ap}}(T_1^{k_1} T_2^{k_2} ... T_n^{k_n}) \right\} \\
&= \frac{C}{q^{k_1+k_2+...+k_n}} \max \left\{ \left| z_1^{k_1} z_2^{k_2} ... z_n^{k_n} \right| : (z_1, z_2, ..., z_n) \in \sigma_\pi(T_1, T_2, ..., T_n) \right\} \\
&\geq C \frac{|\lambda_i|^{k_i}}{q^{k_i}} \left(\prod_{\substack{j=1 \\ j \neq i}}^{n} \frac{\left|\mu_j^{(i)}\right|^{k_j}}{q^{k_i}} \right), \text{ for all } (k_1, k_2, ..., k_n) \in \mathbb{Z}_+^n.
\end{aligned} \tag{3.6}$$

Since $|\lambda_i| > q$, from (3.6) we obtain that, $\lim_{k_i \to \infty} \left\| T_1^{k_1} T_2^{k_2} \dots T_n^{k_n} x \right\| = \infty$, for all $k_j \in \mathbb{Z}_+$, $j \neq i$. And this holds for every $1 \leq i \leq n$. □

Before we state some corollaries of Theorem 4, we'll give one simple example.

Example 3.1. Let $\{e_n : n \in \mathbb{N}\}$ be the canonical base of $\ell^1 \equiv \ell^1(\mathbb{N})$ and $B : \ell^1 \to \ell^1$ be the backward shift,

$$Be_n = \begin{cases} 0, & \text{if } n = 1 \\ e_{n-1}, & \text{if } n \geq 1 \end{cases}, \quad n \in \mathbb{N}.$$

For this operator (see, for example [5, Corollary 6.6]),

$$\sigma_{\text{p}}(B) = \{\lambda \in \mathbb{C} : |\lambda| < 1\},$$

$$\text{Ker}(B - \lambda) = \{\alpha(1, \lambda, \lambda^2, \dots) : \alpha \in \mathbb{C}\}, \text{ for every } \lambda \in \sigma_{\text{p}}(B),$$

$$\sigma(B) = \{\lambda \in C : |\lambda| \leq 1\} = \sigma_{\text{ap}}(B).$$

Let $a_1, a_2, ..., a_n \in \mathbb{R}$ and $\lambda_0 \in \mathbb{C}$ are such that

$$1 < |\lambda_0|^{-1} < a_1 < a_2 < \dots < a_n, \tag{3.7}$$

and let

$$T_i = a_i B, \ 1 \leq i \leq n.$$

It can be easily verified, directly or by applying the spectral mapping theorems for the spectrum and the approximate point spectrum (the later one can be regarded as a special case of Theorem 1 for one operator and the polynomials $p_i : \mathbb{C} \to \mathbb{C}$ defined with $p_i(z) = a_i z$, $1 \leq i \leq n$) that

$$\sigma(T_i) = \sigma(a_i B) = \{\lambda \in \mathbb{C} : |\lambda| \leq a_i\} = \sigma_{\text{ap}}(a_i B),$$

and

$$\sigma_{\text{p}}(T_i) = \sigma_{\text{p}}(a_i B) = \{\lambda \in \mathbb{C} : |\lambda| < a_i\},$$

for all $1 \leq i \leq n$.

Clearly, $T_1, T_2, ..., T_n$ are pairwise commuting operators. But, none of these operators is bounded below (for example, $\|T_i e_1\| = 0 < C \|e_1\|$, for all $C > 0$ and $1 \leq i \leq n$) and they do not satisfy the condition (P.2) (since $r(T_i) = a_i > 1$, if the operators satisfy (P.2), they will have the same spectral radius, which contradicts (3.7)).

Independently of Theorem 4 we will show that in every open ball in ℓ^1 there is a vector x such that $\mathrm{Orb}(\{T_i\}_{i=1}^n, x)$ tends to infinity.

Let $y = (y_n)_{n\geq 1} \in \ell^1$ and $\varepsilon > 0$. By the choice of λ_0, there is $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that $\sum_{j=n_0+1}^{\infty} |y_j| < \varepsilon/3$ and $\sum_{j=n_0+1}^{\infty} |\lambda_0|^{j-1} < \varepsilon/3$. Let

$$x_{\lambda_0} = (y_1, \dots, y_{n_0}, \lambda_0^{n_0}, \lambda_0^{n_0+1}, \dots) = \sum_{j=1}^{n_0} y_j e_j + \sum_{j=n_0+1}^{\infty} \lambda_0^{j-1} e_j.$$

Then,

$$\|y - x_{\lambda_0}\| = \sum_{j=n_0+1}^{\infty} \left| y_j - \lambda_0^{j-1} \right| \leq \sum_{j=n_0+1}^{\infty} |y_j| + \sum_{j=n_0+1}^{\infty} |\lambda_0|^{j-1} < \varepsilon,$$

and, if $(k_1, k_2, \dots, k_n) \in \mathbb{Z}_+^n$ is such that $k_1 + k_2 + \dots + k_n \geq n_0$,

$$\left\| T_1^{k_1} T_2^{k_2} ... T_n^{k_n} x_{\lambda_0} \right\| = \left\| a_1^{k_1} a_2^{k_2} \dots a_n^{k_n} B^{k_1+k_2+\dots+k_n} x_{\lambda_0} \right\| = a_1^{k_1} a_2^{k_2} ... a_n^{k_n} \frac{|\lambda_0|^{k_1+k_2+\dots+k_n}}{1 - |\lambda_0|}.$$

Since (3.7) implies that $a_i |\lambda_0| > 1$, for all $1 \leq i \leq n$, we have

$$\begin{aligned} \lim_{k_i \to \infty} \left\| T_1^{k_1} T_2^{k_2} \dots T_n^{k_n} x_{\lambda_0} \right\| &= \left[\frac{1}{1-|\lambda_0|} \prod_{\substack{j=1 \\ j\neq i}}^{n} \left(a_j |\lambda_0| \right)^{k_j} \right] \lim_{k_i \to \infty} \left(a_i |\lambda_0| \right)^{k_i} \\ &= \infty, \text{ for all } k_1, \dots, k_{i-i}, k_{i+i} \dots, k_n \in \mathbb{Z}_+ . \end{aligned}$$

Remark 3.1: For the vector x_{λ_0} in the previous example $\mathrm{Orb}(a_i B, x_{\lambda_0})$ tends to infinity for every $1 \leq i \leq n$. But the operators $a_1 B, a_2 B, \dots, a_n B$ do not share the same set of vectors such that each one of them has an orbit tending to infinity under each of the operators. For example, if $\mu \in \mathbb{C}$ is such that $a_1 \leq |\mu|^{-1} < a_2$, and x_μ is the vector constructed in a similar way as x_{λ_0}, i.e.

$$x_\mu = (y_1, \dots, y_{n_1}, \mu^{n_1}, \mu^{n_1+1}, \dots) = \sum_{j=1}^{n_1} y_j e_j + \sum_{j=n_1+1}^{\infty} \mu^{j-1} e_j,$$

for some sufficiently large n_1, then $\mathrm{Orb}(\{a_i B\}_{i=2}^n, x_\mu)$ and, consequently $\mathrm{Orb}(a_i B, x_\mu)$, will tend to infinity, for each $2 \leq i \leq n$. But $\mathrm{Orb}(a_1 B, x)$ does not tend to infinity:

$$\left\| T_1^{k_1} x_\mu \right\| = a_1^{k_1} \left\| B^{k_1} x_\mu \right\| = \frac{a_1^{k_1} |\mu|^{k_1}}{1 - |\mu|}, \text{ for all } k_1 > n_1,$$

and consequently, since $a_1 \leq |\mu|^{-1}$,

$$\lim_{k_1 \to \infty} \left\| T_1^{k_1} x_\mu \right\| = \lim_{k_1 \to \infty} \frac{a_1^{k_1} |\mu|^{k_1}}{1 - |\mu|} = \begin{cases} \frac{1}{1-|\mu|}, & \text{if } a_1 = |\mu|^{-1} \\ 0, & \text{if } a_1 < |\mu|^{-1}. \end{cases}$$

Remark 3.2: $T \in B(X)$ is *hypercyclic operator* if there is a vector $x \in X$ such that $\mathrm{Orb}(T, x)$ is dense in X. The vector x with this property is said to be *hypercyclic vector* for T. If T is hypercyclic operator, then the set of all hypercyclic vectors for T is dense G_δ set in X ([2, Lemma III.5.1], [13, Theorem V.38.2]). By definition, the n-tuple of pairwise commuting operators $\mathbf{T} = (T_1, T_2, ..., T_n)$ is a *hypercyclic* *n-tuple* if the there is a vector $x \in X$ such that $\mathrm{Orb}(\{T_i\}_{i=1}^n, x)$ is dense in X ([6]). If at least one of the operators $T_1, T_2, ..., T_n$ is hypercyclic, or the semigroup generated by $T_1, T_2, ..., T_n$ i.e., $\mathcal{T} = \left\{T_1^{k_1} T_2^{k_2} ... T_n^{k_n} : (k_1, k_2, ..., k_n) \in \mathbb{Z}_+^n\right\}$, contains a hypercyclic operator S (which may occur even if none of the operators $T_1, T_2, \ldots, T_n$ is hypercyclic, a simple example will be $\mathbf{T} = (2I, 2^{-1}B)$, where I is the identity operator, B the backward shift on ℓ^1 and $S = (2I)^2 \cdot (2^{-1}B) = 2B$), then the n-tuple $\mathbf{T} = (T_1, T_2, \ldots, T_n)$ is hypercyclic ([6, Proposition 2.1]). By [14, Theorem 1], each of the operators $T_i = a_i B$, $1 \leq i \leq n$, in Example 3.1 hypercyclic. Hence $\mathbf{T} = (T_1, T_2, ..., T_n)$ is a hypercyclic n-tuple and ℓ^1 will contain at least one dense G_δ set of vectors x such that,

$$\mathrm{Orb}(\{a_i B\}_{i=1}^n, x) = \left\{a_1^{k_1} a_2^{k_2} \ldots a_n^{k_n} B^{k_1+k_2+\ldots+k_n} x : (k_1, k_2, \ldots, k_n) \in \mathbb{Z}_+^n\right\},$$

is dense in ℓ^1.

Corollary 4.1. *If $\mathbf{T} = (T_1, T_2, \ldots, T_n)$ is an n-tuple of pairwise commuting operators on an infinite dimensional complex Banach space X such that $r(T_i) > 1$ for all $1 \leq i \leq n$, then there is a dense set $D_1^* \subset X^*$ such that the $n-$tuple orbit $\mathrm{Orb}(\{T_i^*\}_{i=1}^n, x^*)$ tends to infinity for every $x^* \in D_1^*$.*

Proof. If $T_1, T_2, ..., T_n$ are pairwise commuting operators on X, so will be their Banach space adjoints $T_1^*, T_2^*, ..., T_n^* \in B(X^*)$. Having in mind that T^* has the same spectrum as T and hence, $r(T^*) = r(T)$, the conclusion follows by Theorem 4. □

Corollary 4.2. *If $\mathbf{T} = (T_1, T_2, \ldots, T_n)$ is an n-tuple of pairwise commuting invertible operators on an infinite dimensional complex Banach space X such that,*

$$\{\lambda \in \mathbb{C} : |\lambda| > 1\} \cap \sigma(T_i) \neq \emptyset \neq \{\lambda \in \mathbb{C} : |\lambda| < 1\} \cap \sigma(T_i), \tag{3.8}$$

for all $1 \leq i \leq n$, then there is a dense set $D_1^{(1)} \subset X$ such that the $2n$-tuple orbit $\mathrm{Orb}(\{T_i\}_{i=1}^n \cup \{T_i^{-1}\}_{i=1}^n, x)$ tends to infinity, for every $x \in D_1^{(1)}$.

Proof. If $T_1, T_2, ..., T_n$ are pairwise commuting invertible operators on X, then $T_1^{-1}, T_2^{-1}, ..., T_n^{-1}$ will also pairwise commute and

$$T_i T_j^{-1} = T_j^{-1} T_j T_i T_j^{-1} = T_j^{-1} T_i T_j T_j^{-1} = T_j^{-1} T_i,$$

for all $i, j \in \{1, 2, ..., n\}$. Since $\sigma(T^{-1}) = \{\lambda^{-1} : \lambda \in \sigma(T)\}$ for every invertible operator $T \in B(X)$, if $T_1, T_2, ..., T_n$ satisfy the conditions in (3.8), then $r(T_i) > 1$ and $r(T_i^{-1}) > 1$, for all $1 \leq i \leq n$, and the conclusion follows from Theorem 4. □

Remark 3.3: Every invertible operator $T \in B(X)$ is bounded below:

$$\|Tx\| \geq \left\|T^{-1}\right\|^{-1} \|x\|, \text{ for every } x \in X,$$

Hence, if $T_1, T_2, ..., T_n$ are pairwise commuting invertible operators, then the operators $T_1, T_2, ..., T_n, T_1^{-1}, T_2^{-1}, ..., T_n^{-1}$ will satisfy the condition (P.1). If, in addition, the operators satisfy the conditions in (3.8), then the conclusion in Corollary 4.2 can be derived from [7, Theorem 2.2].

In the next two corollaries we assume that T^* denotes the Hilbert space adjoint of the operator T.

Corollary 4.3. *If $\mathbf{T} = (T_1, T_2, ..., T_n)$ is an n-tuple of pairwise commuting operators on an infinite dimensional complex Hilbert space H such that $r(T_i) > 1$ for all $1 \leq i \leq n$, then there is a dense set $D_1^{(2)} \subset H$ such that the n-tuple orbit* $\mathrm{Orb}(\{T_i^*\}_{i=1}^n, x)$ *tends to infinity for every $x \in D_1^{(2)}$.*

Proof. If $T_1, T_2, ..., T_n$ are pairwise commuting operators on H, then the corresponding Hilbert space adjoints $T_1^*, T_2^*, ..., T_n^* \in B(H)$ will also commute pairwise. Since the spectrum of a Hilbert space adjoint T^* of an operator $T \in B(H)$ satisfies $\sigma(T^*) = \{\bar{\lambda} : \lambda \in \sigma(T)\}$ and hence, $r(T^*) = r(T)$, the conclusion follows by Theorem 4. □

Corollary 4.4. *If $\mathbf{T} = (T_1, T_2, ..., T_n)$ is an n-tuple of pairwise commuting normal operators on an infinite dimensional complex Hilbert space H such that $r(T_i) > 1$ for all $1 \leq i \leq n$, then there is a dense set $D_1^{(3)} \subset H$ such that the $2n$-tuple orbit* $\mathrm{Orb}(\{T_i\}_{i=1}^n \cup \{T_i^*\}_{i=1}^n, x)$ *tends to infinity for every $x \in D_1^{(3)}$.*

Proof. If $T_1, T_2, ..., T_n$ are pairwise commuting normal operators on H then, by the Fuglede-Putnam theorem $T_1, T_2, ..., T_n, T_1^*, T_2^*, ..., T_n^*$ will be pairwise commuting normal operators on X. Since $r(T^*) = \|T^*\| = \|T\| = r(T)$ for every normal operator $T \in B(H)$, the conclusion follows from Theorem 4. □

4. N-TUPLE WEAK ORBITS TENDING TO INFINITY

In the section we are going to give only the corresponding result of Theorem 4 for n-tuple weak orbits.

Theorem 5. *If $\mathbf{T} = (T_1, T_2, ..., T_n)$ is an n-tuple of pairwise commuting operators on an infinite dimensional complex Banach space X such that $r(T_i) > 1$ for all $1 \leq i \leq n$, then there is a dense set $D_2 \subset X \times X^*$ such that the n-tuple weak orbit* $\mathrm{Orb}(\{T_i\}_{i=1}^n, x, x^*)$ *tends to infinity for every $(x, x^*) \in D_2$.*

Proof. Let $(x_0, x_0^*) \in X \times X^*$ and $\varepsilon > 0$. If $\lambda_1, \lambda_2, ..., \lambda_n \in \mathbb{C}$ are as in the proof of Theorem 4, let $q \in \mathbb{R}$ and $C > 0$ are such that,

$$1 < q < q^2 < \min\{|\lambda_1|, |\lambda_2|, ..., |\lambda_n|\},$$

and

$$C\left(\frac{q}{q-1}\right)^n < \frac{\varepsilon}{2^{1/p}},$$

assuming that $p = \infty$ if the norm on $X \times X^*$ is the max-norm. Now, let

$$a_{g(k_1,k_2,...,k_n)} = \frac{C^2}{q^{2(k_1+k_2+...+k_n)}} > 0, \text{ for } (k_1, k_2, ..., k_n) \in \mathbb{Z}_+^n,$$

where $g : \mathbb{Z}_+^n \to \mathbb{Z}_+$ is as in the proof of Theorem 4. Then

$$\sum_{(k_1,k_2,\ldots,k_n)\in\mathbb{Z}_+^n} a^{1/2}_{g(k_1,k_2,\ldots,k_n)} = \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\cdots\sum_{k_n=0}^{\infty}\frac{C}{q^{k_1+k_2+\ldots+k_n}} = C\left(\frac{q}{q-1}\right)^n < \frac{\varepsilon}{2^{1/p}},$$

and, by Theorem 3, there are $x \in X$ and $x^* \in X^*$ such that,

$$\|x - x_0\| < \frac{\varepsilon}{2^{1/p}},\ \|x^* - x_0^*\| < \frac{\varepsilon}{2^{1/p}}, \tag{4.1}$$

and

$$\left|\left\langle T_1^{k_1}T_2^{k_2}\ldots T_n^{k_n}x, x^*\right\rangle\right| \geq \frac{C^2}{q^{2(k_1+k_2+\ldots+k_n)}}\left\|T_1^{k_1}T_2^{k_2}\ldots T_n^{k_n}\right\|, \tag{4.2}$$

for all $(k_1, k_2, ..., k_n) \in \mathbb{Z}_+^n$. By (4.1), in both cases, $1 \leq p < \infty$ and $p = \infty$, we have

$$\|(x, x^*) - (x_0, x_0^*)\|_p = \|(x - x_0, x^* - x_0^*)\|_p < \varepsilon,$$

and, if $\mu_1^{(i)}, ..., \mu_{i-1}^{(i)}, \mu_{i+1}^{(i)}, ..., \mu_n^{(i)} \in \mathbb{C}$ are as in the proof of Theorem 4, by (4.2) we have

$$\left|\left\langle T_1^{k_1}T_2^{k_2}\ldots T_n^{k_n}x, x^*\right\rangle\right| \geq C\frac{|\lambda_i|^{k_i}}{q^{2k_i}}\left(\prod_{\substack{j=1\\ j\neq i}}^{n}\frac{\left|\mu_j^{(i)}\right|^{k_j}}{q^{2k_i}}\right) \tag{4.3}$$

for all $(k_1, k_2, ..., k_n) \in \mathbb{Z}_+^n$. Since $|\lambda_i| > q^2$, from (4.3) we obtain that, for every $1 \leq i \leq n$, $\lim_{k_i\to\infty}\left|\left\langle T_1^{k_1}T_2^{k_2}\ldots T_n^{k_n}x, x^*\right\rangle\right| = \infty$, for all $k_j \in \mathbb{Z}_+$, $j \neq i$. □

REFERENCES

[1] H. Baklauti, K. Feki, *On joint spectral radius of commuting operators in Hilbert spaces*, Linear Algebra Appl. Vol. 557 (2018), 455-463.

[2] B. Beauzamy, *Introduction to operator theory and invariant subspaces*, North Holland Math. Library 47, North Holland, Amsterdam, 1988

[3] M.-D. Choi, C. Davis, *The spectral mapping theorem for joint approximate point spectrum*, Bull. Amer. Math. Soc., Vol. 80, No.2 (1974), 317-321.

[4] M. Chō, W. Żelazko, *On geometric spectral radius of commuting n-tuples of operators*, Hokkaido Math. J. 21 (2) (1992), 251-258.

[5] J. B. Conway, *A Course in Functional Analysis*, Springer-Verlag Inc., New York, 1985

[6] N.S. Feldman, *Hypercyclic tuples of operators and somewhere dense orbits*, J. Math. Appl., 346 (2008), 82-98.

[7] S. Mančevska, M. Orovčanec, *N-Tuple Orbits tending to infinity*, Proceedings of the CODEMA 2020 (2021), 24-31.

[8] S. Mančevska, M. Orovčanec, *N-Tuple Weak Orbits Tending to Infinity for Banach Space Operators*, Proceedings of the First Western Balkan Conference on Mathematics and Applications, 10-12-June 2021, Prishtine, Kosovo (2021), 76-83.

[9] S. Mančevska, *N-Tuple Orbits tending to infinity for Hilbert space operators*, Mat. Bilten 45. No. 2 (2021), 143-148.

[10] S. Mančevska, M. Orovčanec, *N-Tuple Weak Orbits Tending to Infinity for Hilbert Space Operators*, Proceedings of the CODEMA 2022 (2023), 27-34.

[11] S. Mančevska, M. Orovčanec, *Orbits tending to infinity under sequences of operators on Banach spaces II*, Math. Maced., Vol. 5 (2007), 57-61.

[12] V. Müller, A. Soltysiak, *Spectral radius formula for commuting Hilbert space operators*, Studia Math. 103 (1992) 329-333.

[13] V. Müller, *Spectral theory of linear operators and spectral systems in Banach algebras*, (2nd ed.), Operator Theory: Advances and Applications Vol. 139, Birkhäuser Verlag AG, Basel - Boston - Berlin, 2007

[14] S. Rolewicz, *On orbits of elements*, Stud. Math., 32 (1969), 17-22.

[15] A. Tajmouati, Y. Zahouan, *Orbit of tuple of operators tending to infinity*, International Journal of Pure and Applied Mathematics Vol. 110, No. 4 (2016), 651-656.

SONJA MANČEVSKA
UNIVERSITY "ST. KLIMENT OHRIDSKI"
FACULTY OF INFORMATION AND COMMUNICATION TECHNOLOGIES,
STUDENTSKA B.B., BITOLA, NORTH MACEDONIA
Email address: sonja.manchevska@uklo.edu.mk

MARIJA OROVČANEC
UNIVERSITY OF SS. CYRIL AND METHODIUS IN SKOPJE,
FACULTY OF NATURAL SCIENCES AND MATHEMATICS,
ARHIMEDOVA 3, SKOPJE, NORTH MACEDONIA
Email address: marijaor@pmf.ukim.mk

Received 4.9.2023
Revised 16.10.2023
Accepted 16.10.2023