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FACULTY OF INFORMATION AND COMMUNICATION TECHNOLOGIES - BITOLA
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Introduction

The International conference on Applied Internet and Information Technologies is a traditional meeting held every year, that sprouts out of collaboration between the University of Novi Sad, Technical Faculty “Mihajlo Pupin”, Zrenjanin, Serbia and the University “St. Kliment Ohridski”, Faculty of Information and Communication Technologies - Bitola, Republic of North Macedonia. The XIII AIIT2023 was held in Bitola, Macedonia on which besides the participants from Serbia and Macedonia there were researchers from Croatia, Bosnia and Herzegovina, Hungary, Finland, Russia, Turkey, Egypt, India and Australia whose contribution was either as authors or as reviewers of the papers.

At the Conference were presented innovative findings in the field of information systems, communications and computer networks, software engineering and applications, data science and big data technologies, artificial intelligence, intelligent systems, business intelligence and IT support to decision-making, data and system security, distributed systems, Internet of Things and smart systems, embedded systems, computer graphics, IT management, e-commerce, e-government, e-education, Internet marketing, and IT practice and experience.

The Conference chairs would like to express gratitude to the authors for their contributions and to express special gratitude to the reviewers for their tremendous work done for selecting the papers with their valuable comments and suggestions that contributed to improve the quality of the papers. Out of more than 60 submitted papers, 51 were selected, presented at the Conference and are published in this proceedings.

The work during the conference was organized in nine sessions: plenary session, five in-person oral sessions, one video session and two poster sessions. During the conference, a round table with participants from academic organizations and IT industry was successfully organized. The theme of the discussions at the round table was "Strengthening the capacities of Faculty of ICT for the realization of strategic cooperation with companies from the IT industry".

AIIT 2023 was very successful conference with fruitful exchange of experiences among the participants reviving the hope of further strengthening a friendly environment after the pandemic crisis. We hope that we will continue with the contribution to the further deepening the development of Internet and information technologies research.

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Finding the Eigenspaces of a Matrix with GeoGebra

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Abstract:
In this paper we present the design of two GeoGebra applets for solving problems involving eigenspaces of a given matrix. One applet features the steps of the standard procedure for finding the eigenvalues and the corresponding linearly independent eigenvectors which will span the eigenspaces. As an additional result, this applet gives an answer to the question whether a given matrix can be diagonalized (by displaying its modal and spectral matrix), or not. The other applet is based on the GeoGebra's commands for the eigenvalues and the Jordan canonical form of a given matrix. It automates the extraction from the similarity matrix in the Jordan decomposition the basis for each of the eigenspace.

Keywords:
Matrices, diagonalizable matrices, defective matrices, eigenvalues, eigenvectors, eigenspaces, Jordan canonical form, Jordan decomposition, GeoGebra

1. Introduction

The standard procedure for solving problems related with the eigenspaces of a given $n \times n$ matrix A involves two steps. The first one is to determine the set of all eigenvalues i.e., the spectrum of A, which comes down to finding all the roots of its characteristic polynomial $P(\lambda) = \det(\lambda I_n - A)$. If λ is an eigenvalue for A, the next step is to determine the eigenvectors corresponding to λ, by solving the matrix equation $A\mathbf{x} = \lambda\mathbf{x}$, with $\mathbf{x} \in \mathbb{C}^n$ as an unknown vector. The eigenspace $E(\lambda)$ corresponding to λ is the set of all solutions of this equation and it is expressed in a form of subspace of $\mathbb{C}^n$ spanned by a finite set of linearly independent eigenvectors for λ. Depending on the size of the matrix and the values of its entries, the calculations that should be performed for each step can be quite extensive and can be properly handled if a suitable software is employed.

GeoGebra is an educational software that includes a computer algebra system which is sophisticated enough to be successfully used for solving various problems when dealing with matrices with moderate sizes. In most of the cases, the CAS specific commands for the eigenvalues and the eigenvectors will display the output in just a few seconds. But, in some cases, the command for the eigenvectors will display a question mark as an output although, for the same matrix, the command for the eigenvalues and the Jordan decomposition will display appropriate results. This situation occurs whenever the matrix is defective i.e., when the matrix is not diagonalizable. A priori, we cannot determine whether a given matrix is defective or not. This can be verified only after the comparison of the algebraic and geometric multiplicities of its eigenvalues. In a case of a defective matrix, we can proceed in two ways. One way is to use other GeoGebra's commands for performing each of the steps of the standard procedure. An applet that is based on this procedure is described in the next section. The other way is to use the result obtained via the Jordan decomposition of the matrix and extract from it all the required information about its eigenvectors and eigenspaces. An applet that automates the extraction of the bases for the eigenspaces is presented in the third section.

The theoretical backgrounds of the spectral theory of matrices, along with variety of solved problems involving the calculation of spectrum and the corresponding eigenvectors and eigenspaces, are usually covered in most of the undergraduate textbooks in linear algebra or operator theory, and their application. For textbooks and articles that also cover the concepts of Jordan canonical form, generalized eigenvectors, and Jordan chains as well, we refer to [1] – [10].

2. GeoGebra applet for the standard procedure

The first applet is based on the output of the **Factors( )** command and works *only if* this command displays a proper factorization of the characteristic polynomial. This means that if the factorization of this polynomial is of form

$$P(x) = (x-\lambda_1)^{k_1} \dots (x-\lambda_m)^{k_m}(x-\mu_1)^{s_1}(x-\bar{\mu}_1)^{s_1} \dots (x-\mu_r)^{s_r}(x-\bar{\mu}_r)^{s_r}, \tag{1}$$

where $\lambda_i \in \mathbb{R}$, for every $i \in \{1, \dots m\}$, $\mu_j \in \mathbb{C}\backslash\{\mathbb{R}\}$ and $\bar{\mu}_j$ denotes the complex conjugate of μ_j, for every $j \in \{1, \dots r\}$, then the **Factors( )** command should display the corresponding factorization in form

$$\begin{pmatrix} x-\lambda_1 & k_1 \\ \vdots & \vdots \\ x-\lambda_m & k_m \\ x^2+p_1x+q_1 & s_1 \\ \vdots & \vdots \\ x^2+p_rx+q_r & s_r \end{pmatrix}, \tag{2}$$

where $x^2+p_1x+q_1 = (x-\mu_j)(x-\bar{\mu}_j),\ j \in \{1, \dots r\}$. Otherwise the applet will not work properly.

Applets based on the other GeoGebra's commands for location of the roots of the characteristic polynomial and that have a different approach to the solution of the corresponding matrix equation $A\mathbf{x} = \lambda\mathbf{x}$, are available through the links in [11] – [13].

The applet is built via the GeoGebra's CAS view. The list of inputs in the CAS cells is given in Table 1, Table 2, and Table 3.

In each table, at the end of some of the inputs a semicolon is placed. This will suppress the output in the corresponding CAS cell and declutter the applet. If necessary, any such semicolon can be omitted.

The matrix must be defined at the beginning. Once the applet is built, it can be used for a different matrix by simply changing the input in the first CAS cell.

Table 1:
List of the inputs in the CAS cells $1 through $11

CAS cell	Input
$1	A:= (*definition of*)
$2	n:=Length(A)
$3	I:=Identity(n)
$4	λ*I-A
$5	P(λ):=Determinant(λ*I-A)
$6	CFactor(P(x))
$7	U:=Factors(P(x))
$8	U_1:Sequence(If(Degree(Element(U,k,1))=1, Element(U, k),Sequence({x - Element(CSolutions(Element(U,k,1)), m), Element(U,k,2)}, m, 1, 2)), k, 1, Length(U));
$9	L:=Flatten(U_1);
$10	AM:Sequence(Flatten({RightSide(CSolve(Element(L, 2*p-1))), Element(L,2*p)}), p, 1, (1/2) Length(L))
$11	GM:=Sequence({Element(AM, q, 1), Length(A)-MatrixRank(A-Element(AM, q, 1)*I)}, q, 1, (1/2) Length(L))

The outputs in CAS cell $10 and CAS cell $11 are in a form of matrices, each with two columns and the same number of rows. The first column is the same for both matrices and it consists of the roots of the characteristic polynomial. The second column in the output of CAS cell $10 displays the *algebraic multiplicity* of the corresponding roots. If all the roots of the characteristic polynomial are real, this matrix will be equal to the one in CAS cell $7. The second column in the output of CAS cell $11 displays the *geometric multiplicity* of the corresponding roots. If this matrix equals to the one in the output of the CAS cell $10, then the matrix A is diagonalizable. Otherwise, the matrix is defective.

Table 2:
List of the inputs in the CAS cells $12 through $24

CAS cell	Input
$12	ListVar:{x1,x2,x3,x4,x5,x6,x7,x8,x9,x10};
$13	X:=Transpose(Take(ListVar, 1, n));
$14	B:=Sequence(Sequence({Element(I, s, t)}, s, 1, Length(I)), t,1,Length(I));
$15	B_1:=Sequence(Flatten(Solve(I* X-Element(B, q), X)),q,1,n);
$16	SYS:=Sequence({"for","λ=", Element(AM, q, 1),":", (A-Element(AM, q, 1)*I),"*", X,"=",0*X}, q, 1, (1/2) Length(L))
$17	SYS1:=Sequence({"for","λ=", Element(AM, q, 1),":", (A-Element(AM, q, 1)*I)* X,"=",0*X}, q, 1, (1/2) Length(L));
$18	SOL1:=Sequence({"for","λ=", Element(AM, q, 1),":",Flatten(Solve((A-Element(AM, q, 1)*I)* X, X))}, q, 1, (1/2) Length(L));
$19	SOL:=Sequence({"for","λ=", Element(AM, q, 1),":",Transpose(RightSide(Element(SOL1, q, 5)))}, q, 1, (1/2) Length(L))
$20	Eigenspaces:=Sequence({"E(", Element(AM, q, 1),")=","span", Sequence(Substitute(Element(SOL, q, 5),Element(B_1, p)), p, 1,n)\{0*X}}, q, 1, (1/2) Length(L))
$21	M_1:=Join(Sequence(Element(Eigenspaces, q, 5), q, 1, (1/2) Length(L)));
$22	M_2:=Sequence(Transpose(Element(M_1, r)), r, 1, Length(M_1));
$23	M:=Transpose(Join(M_2))
$24	If(Element(Dimension(M), 1)==Element(Dimension(M), 2), {{"SpectralMatrix=",Expand(M^(-1)*A*M)}}, nondiagonalizable)

For matrices with dimensions larger than 10×10 the list of unknows in CAS cell $12 should be extended with x11, x12, x13… , as needed.

The eigenspaces will be displayed in the output of the CAS cell $20 in a form of a matrix where, for each different eigenvalue, the corresponding eigenspace is given by its basis.

If the matrix is diagonalizable then its modal matrix will be displayed in CAS cell $23, while its spectral matrix will be displayed in CAS cell $24. If this is the case, for verification, the inputs listed in the next table can be added.

Table 3:
List of the inputs in the CAS cells $25 through $27 (optional)

CAS cell	Input
$25	M*(Expand(M^(-1)*A*M))*M^(-1)
$26	Expand(Eigenvalues(A))
$27	Expand(JordanDiagonalization(A))

The output in the CAS cell $25 should be equal to the matrix defined in CAS cell $1. The elements in the output list in the CAS cell $26 are the eigenvalues of the matrix. Each value is listed as many times as its algebraic multiplicity. The second matrix in the output of the CAS cell $27, *up to a permutation of its diagonal elements*, should equals to one in the output in the CAS cell $25.

Usually, the commands in CAS cell $26 and CAS cell $27 work quite well if the expressions are replaced with Eigenvalues(A) and JordanDiagonalization(A), respectively, but for some matrices we've observed unusually coined expressions in the outputs, which were overcome by adding the external command **Expand( )**.

While the Jordan canonical form of a given square matrix is unique, up to a permutation of its Jordan blocks, the similarity matrix or, in a case of a diagonalizable matrix its modal matrix, is not. This can happen even for the same permutation of the Jordan blocks. This is due to the fact that for each eigenvalue, regardless of whether the matrix diagonalizable or defective, neither the corresponding eigenvectors, nor the basis for corresponding eigenspaces is unique. Hence, the modal matrix in the output of CAS cell \$27, or any other application with similar capacities, may differ from the modal matrix obtained in the output of the CAS cell \$27. This does not affect the results for the eigenspaces themselves. They are just described differently i.e., with a different set of vectors as their basis.

3. Alternative GeoGebra applet for finding the eigenspaces

In some cases, the GeoGebra's command **Factors( )** does not factorize the characteristic polynomial in form (2). For example, instead of trinomials, the matrix in the CAS cell \$7 in the previous applet may contain polynomials of larger degrees. In these cases, one alternative is to use other GeoGebra's commands to locate the roots of the characteristic polynomial. The other one is to use GeoGebra's command **JordanDiagonalization( )** and build the applet with the inputs in the CAS cells as listed in Table 4. Again, the matrix must be defined at the beginning i.e., in the CAS cell \$1.

Table 4:
List of the inputs in the CAS cells for the alternative applet

CAS cell	Input
\$1	A:= (*definition of*)
\$2	I:=Identity(Length(A))
\$3	o:=Transpose(Sequence(0, k, 1, Length(A)))
\$4	E:=Eigenvalues(A)
\$5	JD:=JordanDiagonalization(A)
\$6	S:=Element(JD, 1)
\$7	SS:=Sequence(Sequence({Element(S, p, q)}, p, 1, Length(S)), q,1,Length(S))
\$8	EE:=Unique(E)
\$9	Eigenspaces:=Sequence({"E(", Element(EE, q), ")=", "span", RemoveUndefined(Sequence(If(Expand((A - Element(EE, q) I) Element(SS, k)) == o, Element(SS, k)), k, 1, Length(A)))}, q, 1, Length(EE))

4. Examples

Below are the results obtained with each applet for few matrices. Both applets work for real and complex matrices and matrices with undefined entries (parameters).

Example 4.1. The eigenspaces for the matrix

$$A = \begin{bmatrix} a & 1 & 0 & 0 \\ 0 & b & 1 & 0 \\ 0 & 0 & a & 1 \\ 0 & 0 & 0 & b \end{bmatrix},$$

obtained with the applets are given in Figure 1 and 2.

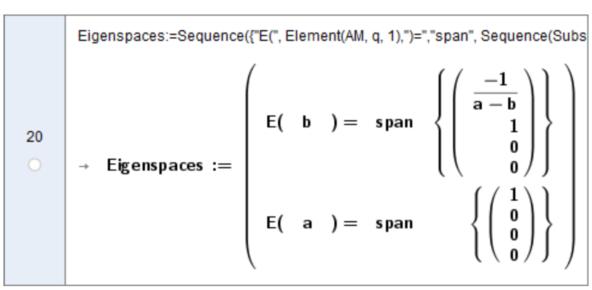

Figure 1: The result for the matrix from the Example 4.1 obtained with the first applet

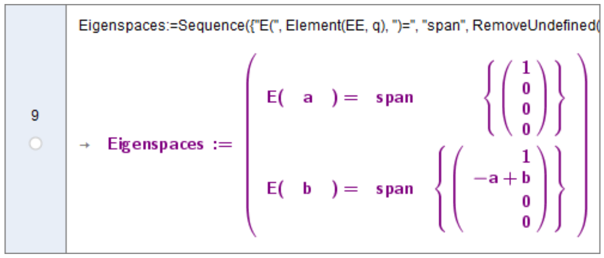

Figure 2: The result for the matrix from the Example 4.1 obtained with the second applet

Example 4.2. The eigenspaces for the matrix

$$A = \begin{bmatrix} 5 & 3 & 5 & 1 & 7 \\ 2 & 3 & 4 & 2 & 4 \\ 0 & -5 & 4 & 7 & 3 \\ 3 & 1 & 6 & 6 & 7 \\ -3 & 2 & -7 & -8 & -7 \end{bmatrix},$$

obtained with the applets are given in Figure 3 and 4.

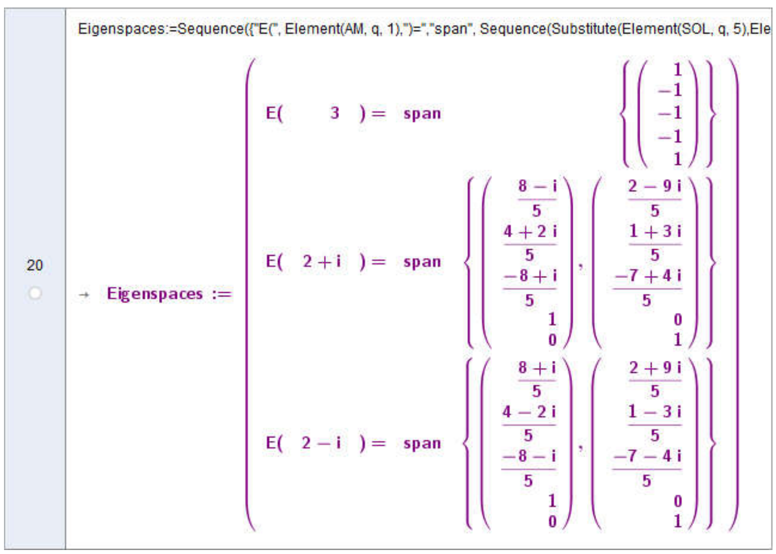

Figure 3: The result for the matrix from the Example 4.2 obtained with the first applet

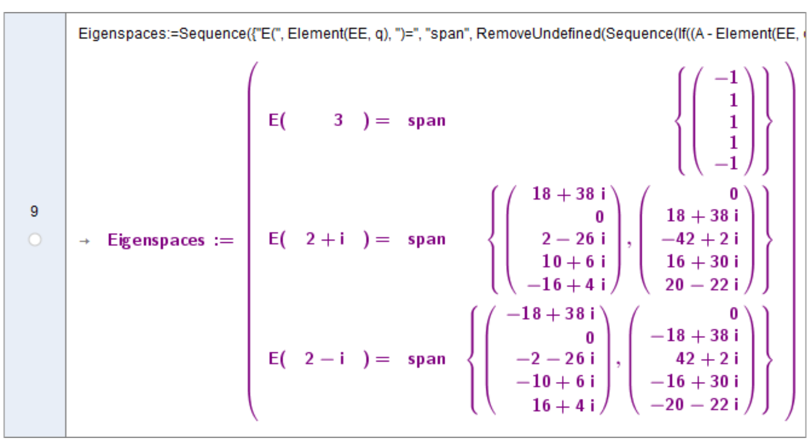

Figure 4: The result for the matrix from the Example 4.2 obtained with the second applet

Example 4.3. The eigenspaces for the matrix

$$A = \begin{bmatrix} 2 & 1 & 3 & 0 & -3 & -2 & -1 & 0 & -2 \\ 0 & 4 & 0 & -1 & 0 & -1 & -2 & 1 & 0 \\ -2 & -13 & 7 & -2 & 0 & -3 & 8 & 1 & -1 \\ -5 & -2 & 5 & 8 & -6 & 0 & 3 & 0 & -3 \\ -2 & -13 & 2 & -2 & 5 & -1 & 8 & 1 & 1 \\ 5 & -1 & -5 & -1 & 7 & 3 & -1 & 0 & -1 \\ 1 & 2 & -1 & -1 & 1 & -1 & 0 & 1 & 0 \\ -2 & -4 & -2 & 5 & 3 & 1 & 2 & 2 & -4 \\ -5 & 1 & 5 & 1 & -7 & 0 & 1 & 0 & 4 \end{bmatrix},$$

obtained with the applets are given in Figure 5, where (a.1) – (a.6) are the results from the first applet, (b.1) – (b.6) are the results from the second applet. The order in which the eigenspaces are displayed in the applets differs from one applet to other.

(a.1)

$$E(5) = \text{span}\left\{ \begin{pmatrix} 1\\0\\1\\0\\0\\0\\0\\0\\0 \end{pmatrix}, \begin{pmatrix} 0\\0\\0\\1\\0\\-1\\0\\0\\1 \end{pmatrix} \right\}$$

(b.1)

$$E(5) = \text{span}\left\{ \begin{pmatrix} 1\\0\\1\\0\\0\\0\\0\\0\\0 \end{pmatrix}, \begin{pmatrix} 0\\0\\0\\1\\0\\-1\\0\\0\\1 \end{pmatrix} \right\}$$

(a.2)

$$E(3) = \text{span}\left\{ \begin{pmatrix} 1\\0\\1\\0\\0\\1\\0\\1\\0 \end{pmatrix} \right\}$$

(b.2)

$$E(3) = \text{span}\left\{ \begin{pmatrix} 1\\0\\1\\0\\0\\1\\0\\1\\0 \end{pmatrix} \right\}$$

(a.3)

$$E(7+i) = \text{span}\left\{ \begin{pmatrix} 0\\0\\-1\\1+i\\-1\\-1\\0\\i\\1 \end{pmatrix} \right\}$$

(b.3)

$$E(7+i) = \text{span}\left\{ \begin{pmatrix} 0\\0\\1\\-1-i\\1\\1\\0\\-i\\-1 \end{pmatrix} \right\}$$

(a.4)

$$E(7-i) = \text{span}\left\{ \begin{pmatrix} 0\\0\\-1\\1-i\\-1\\-1\\0\\-i\\1 \end{pmatrix} \right\}$$

(b.4)

$$E(7-i) = \text{span}\left\{ \begin{pmatrix} 0\\0\\-i\\1+i\\-i\\-i\\0\\1\\i \end{pmatrix} \right\}$$

(a.5)

$$E(2+i) = \text{span}\left\{ \begin{pmatrix} 0\\-i\\-i\\0\\-i\\0\\-i\\1\\0 \end{pmatrix}, \begin{pmatrix} \frac{1-2i}{5}\\ \frac{2-4i}{5}\\ \frac{6-2i}{5}\\ 1\\ \frac{6-2i}{5}\\ -1\\ -i\\ 0\\ 1 \end{pmatrix} \right\}$$

(b.5)

$$E(2+i) = \text{span}\left\{ \begin{pmatrix} -1+2i\\0\\-4-2i\\-5\\-4-2i\\5\\2+i\\4+2i\\-5 \end{pmatrix}, \begin{pmatrix} 0\\-1+2i\\-1+2i\\0\\-1+2i\\0\\-1+2i\\-2-i\\0 \end{pmatrix} \right\}$$

(a.6) (b.6)

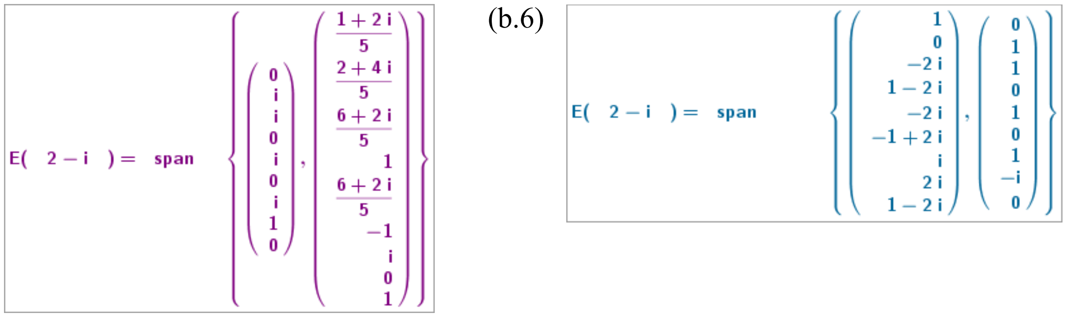

Figure 5: Comparison of the results obtained from the applets for the matrix from the Example 4.1

Example 4.4. The matrix

$$A=\begin{bmatrix} 31 & -7 & 49 & 24 & -5 & 27 & -50 & -33 \\ 4 & -2 & 6 & 3 & -1 & 3 & -6 & -4 \\ -57 & 12 & -87 & -42 & 9 & -47 & 87 & 58 \\ -13 & 0 & -18 & -10 & 3 & -10 & 18 & 13 \\ 16 & -6 & 24 & 11 & -3 & 12 & -24 & -16 \\ 8 & 2 & 11 & 6 & -2 & 6 & -11 & -8 \\ -60 & 15 & -90 & -43 & 9 & -48 & 90 & 61 \\ 31 & -9 & 47 & 22 & -4 & 25 & -48 & -33 \end{bmatrix},$$

has only one eigenvalue and one eigenspace with a dimension equal to 3. Its basis obtained with the applets are shown in Figure 6 and 7.

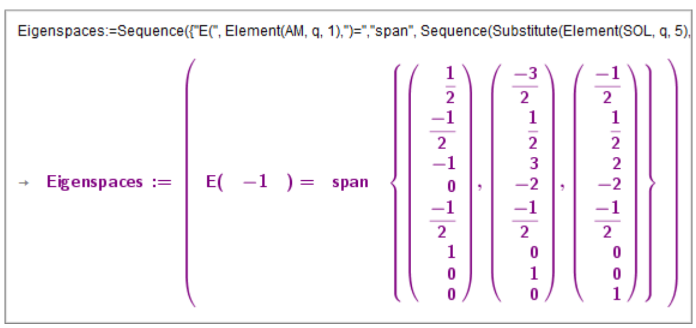

Figure 6: The result for the matrix from the Example 4.3 obtained with the first applet

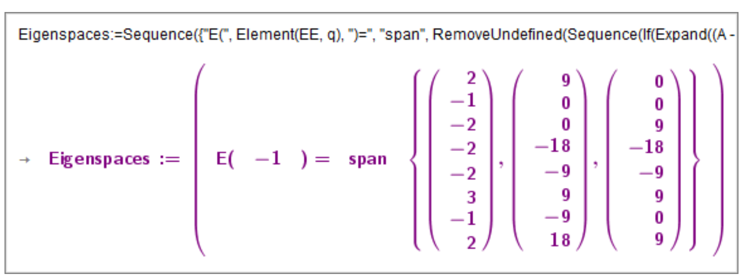

Figure 7: The result for the matrix from the Example 4.3 obtained with the second applet

5. Limitations of the applets

Both applets were tested on various type of matrices, including the ones with undefined entries (like the one from Example 4.1), randomly generated entries, and the ones that were designed with the intention that the output in specific CAS cells can be easily verified with the results obtained if the calculations were performed manually. In some cases, the results were quite impressive. In others, due to the complexity of the algorithms and the extensiveness of the eternal calculation, one or both applets, failed to display the answers. This happened even in the cases where the inputs in some of the CAS cells were simplified so that can be performed step by step across additional CAS cells (number of which depends on the size of the matrix) and the proper results would be obtained quite quickly.

In the cases where the intermediate results contain some sort of rounding, it was more likely that applets will not produce a proper output in the next cells.

Sometimes problems were overcome if the objects were recomputed (via View → Recompute all objects) or reentering the input in one or more cell by simply positioning the cursor at the end of the expression and pressing Enter. In many cases, saving the changes in the applet after modification of the matrix in the CAS cell $1, closing the application and reopening it, resolved the incorrect display in the outputs of the cells. Increasing the CAS Timeout in GeoGebra Classic 5 from the default 5 seconds to the maximum of 60 seconds, was also helpful in many cases.

6. Conclusion

Depending on the matrix, each applet has a capacity to quickly display the eigenvalues, eigenvectors, eigenspaces and additional intermediate results about the matrix. Each one can be quite useful tool in the undergraduate courses in matrix theory. It can used by teachers for demonstration during the lectures and by students when solving problems that involve finding the eigenvalues and corresponding eigenvectors and eigenspaces of a given square matrix. The applets incorporate a proper interpretation of intermediate results and the outputs of the GeoGebra's commands. This can help students to fully grasp the concept of the Jordan decomposition of a matrix and understand the information that this decomposition contains about a given matrix.

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