# MYERSON-SATTERTHWAITE THEOREM AND ASYMMETRIC FPA AUCTIONS

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***Abstract:***

*In this paper Myerson-Satterthwaite theorem with asymmetric First price auction (FPA) has been subject of investigation. Bilateral inefficiency trade theorem versus the efficiency of the FPA auctions in which there is supposedly no dominant strategy, where bids are private information, and are made simultaneously, where highest bid wins and winning bidder pays the winning bid. This type of auction may not be Pareto efficient (this condition requires that the item is allocated to the bidder with highest valuation). But in the sealed FPA auctions highest bidder does not know other bidders' valuations and may lose to another bidder. In the auction setting we set reserve price that causes efficiency loss and decreases probability of trade. The results are ambiguous dependent on the type of the solution method used. Three methods of solution were used: Fixed point finite difference iterations, Backward shooting method, and Constrained strategic equilibrium (C.S.E). The reserve price set was 0.5 since $\theta_s \in (0,1)$and $\theta_b \in (0,1)$, so the buyers' value is likely to be $[0.1,1]$ and the sellers' value is likely to be $[0,0.9]$,so in such case reserve price would eliminate low bidder types. The results are ambiguous in a sense that under Backward shooting method convergence is not true, so the Myerson-Satterthwaite theorem does hold which is not case under Fixed finite difference point iterations, and Constrained strategic equilibrium (C.S.E).Phenomenon known as winner's curse occurs in a case of incomplete information.*

***Key words:*** *FPA, asymmetric auctions, C.S.E, Backward shooting method, Fixed point finite difference iterations, winner's curse*

*JEL classification: D44*

## Introduction

In the Myerson-Satterthwaite setting, people have private information about the utilities for various exchanges of goods at different prices. The Myerson-Satterthwaite theorem (MS) is an important result in mechanism design theory and asymmetric information and this theorem is due to Myerson, Satterthwaite (1983) paper. The main result of the theorem states that there is no efficient way for two parties to trade when they have secret and probabilistically varying CDF's and PDF's, without the risk of one-party trading at loss. Proofs of this theorem are provided in the auction theory graduate textbooks such as Khrisna (2009) and Milgrom(2004).This theory relates back to most famous adverse selection problem posed as the lemons problem ,as Akerlof (1970).Utility function of the seller and the buyer in that model are: $Us = M + \sum_i x_i$(seller's utility) and $U_b = M + \sum_i \frac{3}{2} x_i$, (buyer's utility) i.e.sellers' get one dollar's worth of utility from one quality-unit of used car, and the buyer's get 3/2 dollars' worth of utility from one quality-unit of used car, M is the consumption other goods than that subject of trade (their price is unitary), $x_i$ is the quality of the *ith*used car. The problem is the average quality provided by the sellers which is not equal to price $\mu \neq p$;but the average quality is $\mu = \frac{p}{2}$,since the quality of lemons is $Uni \sim [0,2]$. In that model buyers do not trade since their expected net gain is negative -1/4*p (they cannot trade with expected loss up to quarter of a dollar).Expected value to the buyer is the price he pays times the quality he receives in this case $\frac{3}{2} * \frac{p}{2} = \frac{3}{4} p$ ,while the expected trade gain is expected value minus price $\frac{3}{4} p - p = -\frac{1}{4} p < 0$.If however they are naïve and trade their loss will be $\frac{1}{4} * \frac{3}{2} = \frac{3}{8}$. As in this example the assumptions of M-S theorem are posed: Individual rationality:$U_b, U_s \geq 0$, weak balanced budget (the auctioneer does not subsidize trade). But the Bayesian-Nash equilibrium is not incentive compatible (trade participants namely seller's cheat),$\forall v_b' : U_b(v_B, v_b') \ngeq U_b(v_B, v_b')$ and it is not ex-post Pareto efficient that the item should be given to then one that vales most but here his value is not equal to the expected quality (there are costs of dishonesty).Market produces gains only for sellers and loss only for the buyers, so this trade is not efficient. This the basic motivation of this paper. A typical feature of auctions is the presence of asymmetric information (see Klemperer (1999), Gibbons (1992)), the appropriate concept therefore is Bayesian-Nashequilibrium[1],(seeKajii,A.,Morris,S.(1997), Harsanyi,JohnC.,(1967/1968)).How is this related to Myerson-Satterthwaite theorem? A trade with private preferences (known to him) may demand more favorable terms than he is in truth willing to accept, and such behavior will lessen the gains from trade or will make some to even trade with loss,Rustichini, A.,Satterthwaite, M. A., Williams, S. R., (1994). How is this related to the auction theory? Well, founder of the auction theory is William S. Vickrey with contributions to the literature he made mostly in the 1960's and 1970's, Vickrey (1961,1962,1976). Auctions are type of games where players payoff depends on other's types of market participants, e.g. Akerlof (1970), and this market models where participants have information that affects other player's payoffs are called adverse selection models. Although the treatment of adverse selection in auction theory has history since 1960's, yet the largest part of the auction theory,

[1]A Bayesian Nash equilibrium is defined as a strategy profile that maximizes the expected payoff for each player given their beliefs and given the strategies played by the other players. That is, a strategy profile $\theta$is a Bayesian Nash equilibrium if and only if for every player $i$, keeping the strategies of every other player fixed, strategy $\theta_i$ maximizes the expected payoff of player $i$according to his beliefs. Or in general BNE equilibrium is a Nash equilibrium of a Bayesian game :$Eu_i(s_i|s_{-i},\theta_i) \geq Eu_i(s'_i|s_{-i},\theta_i), \forall s'_i(\theta_i) \in S_i$,where $s_i \in \Theta_i \rightarrow A_i$,and $\theta_i \in \Theta_i$ also utilities are $u_i: A_1 \times A_2 \times \dots\dots..\times A_i \times \theta_1 \times \theta_2 \times \dots\dots.\times \theta_i \rightarrow \mathbb{R}$,where $a_i \in A_i$ denotes finite action set.

puts adverse selection aside to focus on the private values case, in which every type of participants utility depends on its own type. Seminal paper in the literature of asymmetric auctions is written by Maskin,Riley (2000),previously bidders were risk neutral and each bidder has a private valuation different from the others (different cumulative distribution functions and probability density functions), the bidders possess symmetric information, expected payments are functions of their bids, McAfee, McMillan,(1987).But in reality this assumptions seem to be very strict namely the assumption of :risk-neutrality of the bidders, IPV's (independence of the private values of the bidders' about the items value),lack of collusion between the buyers, and especially symmetry of the bidders' beliefs are not describing the reality at best. Relaxing of the risk-neutrality assumption was made in Riley and Samuelson (1981),show that when bidders are risk-averse than seller favors high-bid auction, even if he also exhibits risk aversion.Milgrom, P., Weber,R.J., (1982), relax the assumption of IPV independence, by reporting that if the reservation prices are pairwise positively correlated they show that English auction(is "open" or fully transparent, as the identity of all bidders is disclosed to each other during the auction)exhibits higher revenue than the high-bid auction. And about the third assumption, Graham, D., Marshall, R.,(1987),allow for bidders to collide such as in McAfee, McMillan,(1992).Graham and Marshal (1987) especially propose that bidders' collusions are more likely to happen in the open type auction where bidders' know their identities (they can directly inspect others behavior).But the symmetric beliefs are rejected in this paper. Which means that Revenue equivalence theorem (RET) will not apply here.FPA-First price auction and SPA-Second price auction where winners plays second best price will not exert same revenues. On this topic (optimal auctions) furthermore Myerson (1981), designed Bayesian-optimal mechanism where it makes use of virtual valuations (virtual values are the derivative of the revenue curve). Now back to Myerson-Satterthwaite theorem. The remarkable result of Myerson,Satterthwaite (1983)paper is that it shows that the distributions don't matter and that the failure of efficient trade is general property. Reny and McAfee (1992) show the nature of the distribution of information matters, and McAfee,(1991) showed that continuous quantities can overturn the Myerson-Satterthwaite theorem. And now put it differently Myerson-Satterthwaite theorem says that if one demands ex-ante budget balance, and interim individual rationality than trade cannot be ex-post efficient, Nachbar (2017).Ex-post efficiency occurs only when buyers value is more than sellers values and opposite is not true.Strict Pareto efficienymenas that :$\forall v \forall x', \sum_i v_i(x) \geq \sum_i v_i(x')$.Ex-ante budget balance means that while a third party is allowed to provide a net subsidy for some types of profiles $(\theta_s, \theta_b)$ ,and collect net tax for the others', thethird-party net transfer must be zero over expectations for $v$ -value of the buyers and $s$ value for the seller-post budget balance in the other hand requires, zero net transfers for all $v, s, \forall v, \sum_i p_i(s(v)) = 0$.Ex-interim individual rationality means that no agent losses from participating in the mechanism, and is ex-interim because it holds for every possible valuation of agent *i*.[2]

[2] More so mechanism is ex-interim individually rational if:$\forall i \forall v_i, \mathbb{E}_{(v_{-i}|v_i)} v_i\ (\chi\ (s_i(v_i), s_{-i}(v_{-i}))) - p_i\ (s_i(v_i), s_{-i}(v_{-i})) \geq 0$,And a mechanism is ex-post efficient if $\forall i \forall v_i, v_i\ (\chi\ (s_i(v_i), s_{-i}(v_{-i}))) - p_i\ (s_i(v_i), s_{-i}(v_{-i})) \geq 0_{(v_{-i}|v_i)} v_i\ (\chi\ (s_i(v_i), s_{-i}(v_{-i}))) -$
$p_i\ (s_i(v_i), s_{-i}(v_{-i})) \geq 0$

Theorem 1.1. Revelation principle Myerson,1981

Suppose that $\psi$ was a Bayes-Nash equilibrium of the indirect mechanism $\Gamma$. Then there exists a direct mechanism that is payoff-equivalent and where truthful revelation is an equilibrium.

*Proof:* $\exists\psi$ *and these strategies are equivalent in direct and indirect mechanisms. Direct revelation mechanism is the one where agent reports his preferences truthfully and hence* $M = \Pi_{i\in 1,2,\dots,n} M_i$*(messages)agents type of profiles are :*$\theta \in \{1,2,\dots,n\}$ *and* $\theta \in \Theta$*,the social choice function is* $f: \Theta \to X$ *where outcome* $x \in X$ *and in a message space there is mapping* $g: M \to X$*. Let's notice that if bidder(player)*$i$ *with type* $\theta$ *deviates and reports his other type* $\theta'$ *that that agent earns* $E_{\theta_{-i}} v_i\left(\psi_i(\theta_i')\psi_{-i}(\theta_{-i})\right) = E_{\theta_{-i}} v_i\left(\psi', \psi_{-i}(\theta_{-i})\right)$ *for some* $\psi'$*and we know that (form above said):*

*Equation 1*

$$E_{\theta_{-i}} v_i\left(\psi_i(\theta_i), \psi_{-i}(\theta_{-i})\right) \geq E_{\theta_{-i}} v_i\left(\psi', \psi_{-i}(\theta_{-i})\right)$$

So this last expression is not profitable ■.

Definition Incentive compatibility (Bayesian Incentive compatibility (BIC) )

*A social choice function* $f: \Theta_1 \times \Theta_2 \dots\dots \times \Theta_n \to X$ *is said to be incentive compatible (IC) or truthfully implementable if the Bayesian game (is a game in which the players have incomplete information about the other players) induce by the direct revelation mechanism( is one where each agent is asked toreport his individual preferences, in which case* $M = \Theta$ *and* $f = g$*) or* $\mathcal{D} = \left(\Theta_{i\in N}, f(\cdot)\right)$*has a pure strategy equilibrium(Bayesian-Nash equilibrium)* $s^*(\cdot) = (s_1^*(\cdot), \dots, s_n^*(\cdot))$ *where* $s_i^*(\Theta_i) = \Theta_i$ *and* $\forall\theta_i \in \Theta$ *and* $\forall i \in N$.

Individual rationality (IR) axiom

First we define as in Myerson (1991), *two-person bargaining problem , to consist of a pair* $(F, v)$ *where* $F$ *is a convex subset of* $\mathbf{R}^2$,$v = (v_1, v_2)$ *is a vector in* $\mathbf{R}^2$*and the set* $F \cap \{(x_1, x_2) | x_1 \geq v_1; x_2 \geq v_2\}$ *is non-empty and bounded. Where* $F$ *is a set of feasible payoff allocations and* $v$ *represents the disagreement point.F is a convex means that the players are assumed that will agree on their jointly randomized strategies so that utility allocations* $x = (x_1, x_2)$ *and* $y = (y_1, y_2)$ *are feasible and* $0 \leq \theta \leq 1$ *so that following expected utility allocation applies* $\theta x + (1-\theta)y$ *.Two-players strategic game form is given as:*$\Gamma = \{(1,2), C_1, C_2 m u_1, u_2\}$ *where* $C_1, C_2$ *are used to denote the pure players strategies set.*

Theorem 1.2 Myerson -Satterthwaite

*Theorem Myerson-Satterthwaite: It is not common knowledge that if trade gains exist i.e. the supports of the CDF functions (Cumulative distributions) of traders have non-empty intersections) then no IC(incentive compatibility) and IR (individual rationality)trading mechanism can be ex-post efficient. Proof: A trading mechanism is ex-post efficient if and only if trade occurs whenever* $s \leq b$

*Equation 2*

$$p(s,b) = \begin{cases} 1 \text{ } if \text{ } s \leq b \\ 0 \text{ } if \text{ } s > b \end{cases}.$$

In the previous expression $p(s,b)$ is a probability of trade which takes value 1 if trade occurs and zero if it doesn't. To prove that ex-post efficiency cannot be attained, it is enough to show that inequality $(*)$ in the corollary hence:

*Equation 3*

$$\int_{\underline{b}}^{\overline{b}}\int_{\underline{s}}^{\min(b,\overline{s})}\left[b-\frac{1-F(b)}{f(b)}-s-\frac{F(s)}{f(s)}\right]f(s)f(b)dsdb.$$

Previous expression equals to:

*Equation 4*

$$\int_{\underline{b}}^{\overline{b}}\int_{\underline{s}}^{\min(b,\overline{s})}[bf(b)+F(b)-1]f(s)dsdb-\int_{\underline{b}}^{\overline{b}}\int_{\underline{s}}^{\min(b,\overline{s})}[sf(s)+F(s)]f(b)dsd=$$

$$-\int_{\underline{b}}^{\overline{s}}[1-F(\theta)]F(\theta)d\theta<0,\underline{b}<\overline{s}.$$

Previous result is proof of Myerson-Satterthwaite theorem about trade inefficiency. Some weaker efficiency criterion is Pareto optimality, one may use that criterion if ex-post efficiency does not work■.

Furthermore one mechanism with transfers $t$ , $p(x(s,b),t)$ where $p$ is the probability of trade given $s, b$,seller and buyers ex-post utilities are given as:

*Equation 5*

$$u(s,b)=x(s,b)-s(p,b)$$

$$v(b,s)=bp(s,b)-x(s,b).$$

Both traders are risk neutral and there are no income effects. Payoffs are defined as:

*Equation 6*

$$X(s)=\int_{\underline{b}}^{\overline{b}}x(s,b)f_b(b)db\,;X(b)=\int_{\underline{s}}^{\overline{s}}x(s,b)f_s(s)ds.$$

Probabilities of trade are defined as:

*Equation 7*

$$P(s)=\int_{\underline{b}}^{\overline{b}}p(s,b)f_b(b)db\,;P(b)=\int_{\underline{s}}^{\overline{s}}p(s,b)f_s(s)ds$$

Incentive compatible mechanism is defined as:

*Equation 8*

IC: $U(s)\geq X(s')-P(s');V(b)\geq P(b)-X(b).$

Individually rational mechanism (IR) is given as:

*Equation 9*

$$\forall s\in[\underline{s},\overline{s}]\vee\forall b\in[\underline{b},\overline{b}],U(s)\geq 0\,;V(b)\geq 0.$$

Lemma 1 (Mirrlees,Myerson)

The mechanism is IC compatible if and only if $P(s)$ is increasing and $P(b)$ decreasing, and :

*Equation 10*

$$\begin{cases} U(s) = U(\underline{s}) + \int_{\underline{s}}^{\bar{s}} P(s)(\theta)d\theta \\ V(b) = V(\underline{b}) + \int_{\underline{b}}^{\bar{b}} P(b)(\theta)d\theta \,. \end{cases}$$

Lemma 1 (proof): From previous we know that $U(s') \geq X(s') - s'P(s'); U(s) \geq X(s) - sP(s)$

*Equation 11*

$$\begin{cases} U(s) \geq X(s') - sP(s') = U(s') + (s' - s)P(s'), \\ U(s') \geq X(s) - s'P(s) = U(s) + (s - s')P(s). \end{cases}$$

If we subtract these inequalities will yield:

*Equation 12*

$$(s' - s)P(s) \geq U(s) - U(s') \geq (s' - s)P(s').$$

Now if we take that $s' > s$ implies that $P(s)$ is decreasing, if we divide by $(s' - s)$ and letting $s' \rightarrow s$ yields $\frac{dU(s)}{ds} = -P(s)$ and integrating produces IC(s'). The same is true for the buyer. To prove the IC for the seller it is suffice to show that following applies:

*Equation 13*

$$s[P(s) - P(s')] + [X(s') - X(s)] \leq 0 \, \forall \, s, s' \in [\underline{s}, \bar{s}] \,.$$

Now from previous by substituting for $X(s)$ and $X(s')$ and by using IC(s') the following will yield:

*Equation 14*

$$X(s) = sP(s) + U(\bar{s}) + \int_{\underline{s}}^{\bar{s}} P(\theta)d\theta \,.$$

And following holds , we show that $ss' \in [\underline{s}, \bar{s}]$ :

*Equation 15*

$$0 \geq s[P(s)P(s)] + sP(s) + \int_{s'}^{\bar{s}} P(\theta)d\theta - sP(s)$$
$$- \int_{s}^{\bar{s}} P(\theta)d\theta = (s' - s)P(s') + \int_{s'}^{s} P(\theta)d\theta = \int_{s'}^{s} [P(\theta) - P(s')]d\theta \,.$$

Where in the last expression $\theta$ is unknown parameter i.e. players type. And previous expression holds only because $P(\cdot)$ is decreasing.

Lemma Individual rationality (IR):

IC mechanism is IR if and only if : $U(\bar{s}) \geq 0 \vee V(\underline{b}) \geq 0$ .And following corollary is introduced:

*Corollary*:

*Equation 16*

$$U(\overline{s}) + V(\underline{b}) = \int_{\underline{b}}^{\overline{b}} \int_{\underline{s}}^{\overline{s}} \left[ b - \frac{1 - F(b)}{F(b)} - s - \frac{1 - F(s)}{F(s)} \right] \mathrm{p(s,b)f(s)f(b)dsdb} \geq 0 \,.$$

*Proof* : Since from IC condition we know that the following applies :

*Equation 17*

$$X(s) = sP(s) + U(\overline{s}) + \int_{\underline{s}}^{\overline{s}} P(\theta)\,d\theta \,.$$

And from the corollary:

*Equation 18*

$$\int_{\underline{b}}^{\overline{b}} \int_{\underline{s}}^{\overline{s}} \mathrm{x(s,b)f(s)f(b)dsdb} = \mathrm{U}(\overline{s}) + \int_{\underline{b}}^{\overline{b}} \int_{\underline{s}}^{\overline{s}} \mathrm{sp(s,b)f(s)f(b)dsdb} + \int_{\underline{b}}^{\overline{b}} \int_{\underline{s}}^{\overline{s}} \mathrm{p(s,b)F(s)f(b)dsdb}.$$

The third term in the right side follows that:

*Equation 19*

$$\int_{\underline{b}}^{\overline{b}} \int_{\underline{s}}^{\overline{s}} \mathrm{p(\theta,b)F(s)f(b)d\theta db} = \int_{\underline{b}}^{\overline{b}} \int_{\underline{s}}^{\theta} \mathrm{p(\theta,b)F(s)f(b)d\theta db} = \int_{\underline{s}}^{\overline{s}} \mathrm{p(s,b)F(s)f(b)dsdb} \,.$$

Analogously for the buyer follows that:

*Equation 20*

$$\int_{\underline{b}}^{\overline{b}} \int_{\underline{s}}^{\overline{s}} \mathrm{x(s,b)f(s)f(b)dsdb} = -V(\underline{b}) + \int_{\underline{b}}^{\overline{b}} \int_{\underline{s}}^{\overline{s}} \mathrm{bp(s,b)F(s)f(b)dsdb} - \int_{\underline{b}}^{\overline{b}} \int_{\underline{s}}^{\overline{s}} \mathrm{p(s,b)F(s)}\left(1 - F(\mathrm{b})\right)\mathrm{dsdb}$$

And if we equate the both sides:

*Equation 21*

$$V(\underline{b}) = \int_{\underline{b}}^{\overline{b}} \int_{\underline{s}}^{\overline{s}} \mathrm{p(s,b)F(s)}\left(1 - F(\mathrm{b})\right)\mathrm{dsdb} - \int_{\underline{b}}^{\overline{b}} \int_{\underline{s}}^{\overline{s}} \mathrm{bp(s,b)F(s)f(b)\,dsdb} = \int_{\underline{s}}^{\overline{s}} \mathrm{p(s,b)F(s)\,f(b)dsdb} \,.$$

IR mechanism is proved since $V(\underline{b}) \geq 0$■.

Balanced budget

Transfers $t: \Re^2 \to \Re^2$ between the players are: $t(v) = \{t_1(v), t_2(v)\}$and $t_i(v)$ is the transfer that player $i$ receives. Now we are considering a transfer $t_i(v) = g_i(v_i) - g_j(v_j)$,where :

*Equation 22*

$$g_i(v_i) = \int_{-\infty}^{\infty} v_j d(v_i, v_j) f_j(v_j) dv_j$$

From previous expression clearly transfers balance. Where $d \in \{0,1\}$i.e. do it or not do it eg. public project, $v_{i,j} \in \{-\infty, \infty\}$, and ex-post efficiency required that :

*Equation 23*

$$d^*(v_1, v_2) = \begin{cases} 1, if\ v_1 + v_2 \geq 0 \\ 0, v_1 + v_2 < 0 \end{cases}$$

Also we have to note that :

*Equation 24*

$$v_i \in \arg\max_{\hat{v}_i} v_i d\left(\hat{v}_i, \hat{v}_j\right) + t\left(\hat{v}_i, \hat{v}_j\right)$$

VCG mechanism does not satisfy weak balanced budget condition. Vickrey-Clarke-Groves, auction is named after Vickrey(1961), Clarke (1971), Groves (1973)for their papers that generalized the idea. VCG mechanism is a direct quasi-linear mechanism.

*Equation 25*

$$(\chi, \tilde{v}) = \arg\max_x \sum_i \bar{v}_i(x)$$

$$p_i(\hat{v}) = \sum_{j \neq i} \hat{v}_j\left(\chi(\hat{v}_{-i})\right) - \sum_{j \neq i} \hat{v}_j\left(\chi(\hat{v})\right).$$

And under Groves mechanism we have :

*Equation 26*

$$\chi(\hat{v}) = \arg\max_x \left(\sum_i \hat{v}_i(x)\right) = \arg\max_x \left(\hat{v}_i(x) + \sum_{j \neq i} \hat{v}_j(x)\right).$$

In Groves mechanism price constraint is given as follows:

*Equation 27*

$$p_i(\hat{v}) = h_i(\hat{v}_{i-1}) - \sum_{j \neq i} \hat{v}_j\left(\chi(\hat{v})\right).$$

There budget balance requires :

*Equation 28*

$$t_1(v) + t_2(v) = (v_1 + v_2)d(v_1, v_2) + h_1(v_2) + h_2(v_1) = 0$$

Or

*Equation 29*

$$h_1(v_1) + h_2(v_2) = \begin{cases} -(v_1 + v_2); if\ v_1 + v_2 \geq 0 \\ 0; v_1 + v_2 < 0 \end{cases}$$

But previous cannot happen if $h_i$ is independent of $v_i$ .Solution is in weak incentive compatibility criterion, so truth as dominant strategy in this mechanism is merely a Bayesian equilibrium rather than a dominant strategy. Bayesian incentive compatibility (BIC) is presented as:

*Equation 30*

$$v_i \in arg\max_{\hat{v}_i} E_{v_j}\left[v_i d\left(\hat{v}_i, v_j\right) + t_i\left(\hat{v}_i, v_j\right) \middle| v_i\right]$$

And the budged balance that satisfies than is $t_1(v) + t_2(v) = 0$.One example of mechanism is second degree price discrimination with a continuum types when monopolist practices price discrimination via quantity discounts when there is continuum of types. Quasi-linear utility function is given as:

*Equation 31*

$$U(\theta) \equiv b(x(\theta), \theta) - t(\theta)$$

Where $x$ is the consumption of the good and $t$ is the amount paid to the firm or taxes to the state (for the amount $x$ units in it). Type of the customer is drawn from a distribution:$F(\cdot); (\theta_1, \theta_2), \theta \in (\theta_1, \theta_2)$.Spence-Mirrlees condition holds at :

*Equation 32*

$$\frac{\partial^2 b(x, \theta)}{\partial \theta \partial x} > 0$$

Now $\frac{\partial b}{\partial x} > 0, \forall x > 0$ ,this follows because $\frac{\partial b}{\partial x}$ is an inverse demand curve , an inverse demand curves are positive ,proof is given below:

*Proof:*

Let $\theta_2 > \theta_1$ and now we observe :

*Equation 33*

$$b(x, \theta_2) - b(x, \theta_1) = \int_0^x \frac{\partial b(z, \theta_2)}{\partial x} dz - \int_0^x \frac{\partial b(z, \theta_1)}{\partial x} dz = \int_0^x \left( \int_{\theta_1}^{\theta_2} \frac{\partial b(z, t)}{\partial \theta \partial x} dt \right) dz > 0$$

Previous expression follows form the fact that $x > 0$ and $\theta_2 > \theta_1$.Expected profit in this mechanism with transfers due to Mirrlees (1971) is given as:

*Equation 34*

$$\Pi = \int_{\theta_1}^{\theta_2} \big(t(\theta) - cx(\theta)\big) f(\theta) d\theta$$

In the previous expression $c$ are the marginal costs, and $t$ are the transfers or the amount paid to the firm or the state (taxes). For the scheme to be feasible it must satisfy IR and IC constraints. The IR constraint is simply:$U(\theta) > 0; \forall \theta \in [\theta_1, \theta_2]$ .For the IC constraints we consider two arbitrary consumer types and each one is larger than the other:

*Equation 35*

$$U(\theta_2) \geq b(x(\theta_1), \theta_2) - t(\theta_1)$$

$$U(\theta_1) \geq b(x(\theta_2), \theta_1) - t(\theta_2)$$

Previous two expressions are known as revealed preference (RP) now if we substitute $U(\theta) \equiv b(x(\theta), \theta) - t(\theta)$ for $t(\theta)$ we have:

*Equation 36*

$$U(\theta_2) \geq b(x(\theta_1), \theta_2) - b(x(\theta_1), \theta_1) + U(\theta_1) = U(\theta_1) + \int_{\theta_1}^{\theta_2} \frac{\partial b(x(\theta_1), t)}{\partial \theta_1} dt$$

$$U(\theta_1) \geq b(x(\theta_2), \theta_1) - b(x(\theta_2), \theta_2) + U(\theta_2) = U(\theta_2) + \int_{\theta_1}^{\theta_2} \frac{\partial b(x(\theta_2), t)}{\partial \theta_1} dt$$

If we combine previous two expressions such as:

*Equation 37*

$$\int_{\theta_1}^{\theta_2} \frac{\partial b(x(\theta_2), t)}{\partial \theta_1} dt \geq U(\theta_2) - U(\theta_1) \geq \int_{\theta_1}^{\theta_2} \frac{\partial b(x(\theta_1), t)}{\partial \theta_1} dt$$

Previous implies that :

*Equation 38*

$$\int_{\theta_1}^{\theta_2} \left( \frac{\partial b(x(\theta_2), t)}{\partial \theta_1} - \frac{\partial b(x(\theta_1), t)}{\partial \theta_1} \right) dt \geq 0$$

By the fundamental theorem of calculus :

*Equation 39*

$$\int_{\theta_1}^{\theta_2} \left( \int_{x(\theta_1)}^{x(\theta_2)} \frac{\partial^2 b(z, t)}{\partial \theta \partial x} dz \right) dt \geq 0$$

For the inequality to hold it must be that :$x(\theta_2) > x(\theta_1)$ .This implies that inverse demand curve is positive and monotonic and differentiable ■.

Now if we set some measure $\epsilon$ (endpoint in integration) towards which all points will converge

*Equation 40*

$$\frac{1}{\epsilon} \int_{\theta_1}^{\theta_1 + \epsilon} \frac{\partial b(x(\theta_1 + \epsilon), t)}{\partial \theta_1} dt \geq \frac{U(\theta_2) - U(\theta_1)}{\epsilon} \geq \frac{1}{\epsilon} \int_{\theta_1}^{\theta_1 + \epsilon} \frac{\partial b(x(\theta_1), t)}{\partial \theta_1} dt$$

For previous to hold it must be that $\epsilon > 0$ so this applies as long as $\epsilon \to 0$ . The limit of the left terms is given as:

*Equation 41*

$$\frac{\partial b(x(\theta_1), \theta_1)}{\partial \theta_1} + \int_{\theta_1}^{\theta_1} \frac{\partial^2 b(x(\theta_1), t)}{\partial \theta_1 \partial x} x'(\theta_1) dt = \frac{\partial b(x(\theta_1), \theta_1)}{\partial \theta_1}$$

And $\frac{\partial b(x(\theta_1), \theta_1)}{\partial \theta_1} \geq \lim_{\epsilon \to 0} \frac{U(\theta_1 + \epsilon) - U(\theta_1)}{\epsilon} \geq \frac{\partial b(x(\theta_1), \theta_1)}{\partial \theta_1}$, this means that first derivative of the utility functions is :

*Equation 42*

$$U'(x) = \frac{\partial b(x(\theta_1), \theta_1)}{\partial \theta_1}$$

Previous expression is the utility function without transfers and with transfers utility function is :

*Equation 43*

$$U(\theta) = U_0 + \int_{\theta_1}^{\theta_2} \frac{\partial b(x(t),t)}{\partial \theta} dt$$

Previous means that if the preference tastes are equal there will be no need of transfers and $U(\theta) = U(0)$. But since $\frac{\partial b}{\partial \theta} > 0$ it follows that $U(\theta) > U(\theta_1) \geq 0, \forall \theta$

Extension to Myerson-Satterthwaite theorem

Supposedly when there are many buyers and sellers (not just one buyer and one seller as M-S supposed) inefficiency asymptotically disappears. This is the case of private goods,

Theorem 2. Shapley-Folkman theorem ,Star (1969)

Let $\mathbb{R}^L$ be an $L$ dimensional Euclidean space, the let $A_i$ denote the its convex hull for any $A \subseteq R^n$, now $\emptyset \neq A_i \subseteq R^n, \wedge x \in \operatorname{con} \sum_{i=1}^{n} A_i$,then $\sum_{i=1}^{n} x_i = x, \wedge x \in \operatorname{con} A_i, \forall i, x_i \in A_i$, for at least $n-m$ indices of $i$.Khan and Rath (2013)

***Proof of the theorem, Zhou (1993)***: let $x \in \operatorname{con}(A)$ has a representation $x = \sum_{i=1}^{n} y_i$ and $y_i \in \operatorname{con}(A_i), \forall i$,and let $y_i = \sum_{i=1}^{n} a_{ij} y_{ij}, a_{ij} > 0, \sum_{i=1}^{n} a_{ij} = 1, y_{ij} \in \mathrm{A}_i$.Constructed $z$ vectors are given as: $z = \sum_{i=1}^{n} \sum_{j=1}^{l} b_{ij} y_{ij}, b_{ij} \geqq 0, \wedge (m+n) b_{ij} \geqq 0$.From previous expression $x = \sum_{i=1}^{n} \sum_{j=1}^{l} b_{ij} y_{ij}, b_{ij} = 1, \forall i$.Now,$x = \sum_{j=1}^{l} b_{ij} y_{ij}, x = \sum_{i=1}^{n} x_i, x_i \in \operatorname{con}(A_i), \forall i$.Because there are $(m+n) b_{ij} \geqq 0$ in total, there is at least one $b_{ij} \geqq 0, \forall i$ ,and there are at most $m$ indices $i$ that have more than one $b_{ij} > 0$,and $x_i \in A_i$ for at least $(n-m)$indices $i$. ■

***Preposition** :*

Preposition also here is continuous function (continuity condition) i.e. the condition for continuity as given in Robbin, et al. (1987), where states that $f$ is said to be continuous on $\mathbb{R}^l$ if

*Equation 44*

$$\forall x_0 \in \mathbb{R}^l \forall \epsilon > 0 \ \exists \delta > 0 \ \forall x \in \mathbb{R}^l [|x - x_0| < \delta \ | \Rightarrow f(x) - f(x_0) < \epsilon |]$$

In the previous expression $\epsilon$ is a trimmed price space. Trimmed space is a location parameter class of probability functions that is parametrized by scalar or vector valued parameter $x_0$ which determines distributions or shift of the distribution. Also, if there are $n$ commodities, and a nonnegative orthant $\Theta$ of Euclidean space $E^n$ is introduced is introduced, then the sets $\{x : x \succcurlyeq_c y\}$ and $\{x : y \succcurlyeq_c x\}$ are closed. Here $\succcurlyeq_c$ are preferences of a trader in a pure exchange economy (Starr 1969). The assumption of convexity assumes that if $\succcurlyeq_c y$ , then $\lambda x + (1-\lambda) x \succcurlyeq_c y$, this means that any weighted average or convex combination of $x$ and $y$ is preferred to $y$ , $0 \leq \lambda \leq 1$. Each trader has initial endowment bundle and stars with a positive amount of some good $x_c > 0$.

Convex hull (A convex hull is the smallest polygon that encloses a group of objects, such as points is given as:

*Equation 45*

$$con \equiv \left\{ \sum_{j=1}^{N} \lambda_j p_j : \lambda_j \geq 0, \forall j, \wedge \sum_{j=1}^{N} \lambda_j = 1 \right\}$$

Now if so, $\mu: \Psi \to \mathbb{R}^L$ is an countably additive measure if $\mu(\emptyset) = 0, \wedge \mu(\cup_{n=1}^{\infty} A_n) = \sum_{i=1}^{n} \mu(A_n)$,when $\{A_n\}$ is a sequence of disjoint pairs of sets in $\Psi$. The measure $\mu$ purely atomic if there is scalar measure $\lambda$ such that $\lambda \ll \mu$, and if $\lambda(A) = 0$, for every measurable set for which $|\mu|(A) = 0$.And if there is a sequence $\{E_k\}$ such that $T = \cup_{k=1}^{\infty} E_k, \forall E_k \in \forall \mu_i, i = 1, \dots, m$. This is called an atomic or only positive measure (Bogachev 2007, Aliprantis and Border 2013, Halmos 2013, Hewitt and Stromberg 2013).

Theorem 3 Second Fundamental welfare theorem

Second Fundamental welfare Theorem: Given an economy such as; $\left(\{X_i, \succcurlyeq_i\}_{i=1}^{I}, \{Y_j,\}_{j=1}^{J}, \overline{\omega}\right)$we assume that $\sum_i \omega \gg 0$, allocation is given with $(x*, y*)$ ,and a price vector $p \neq 0$, (here $p = p_1, \dots\dots, p_n$ ), and this combination constitutes price quasi-equilibrium with transfers $(\omega_1, \dots., \omega_I)$ , subject to the following budget constraint: $\sum_i \omega_i = p\overline{\omega} + \sum_i p y_j^*$,where:

1. $\forall j, y_j^*, \max_j p y_j^*, p y_i \leq p y_j^*, \forall y_j \in Y_j$.( firm maximizes profit by producing $y_j^*$)
2. $\forall i, \exists x_i \succ_i x_i^* \to p x_i \geq \omega_i$.(if $x_i \succ x_i^*$ then it cannot cost less than $x_i^*$).
3. And $\sum_i x_i^* = \overline{\omega} + \sum_j y_j^*$.(budget constraint )

Here we define $V_i = \sum_{i=1}^{i} x_i \succ x_i^*$ (strictly preferred than $x_i^*$ ) also $V = \sum V_i$,and $V_i$ is convex due to convexity of a preference relation $\succ_i$,and $V$ is convex since$\forall V_i$ are convex. Point number 2 is the only difference with the definition of price equilibrium since $p x_i \gtrsim \omega_i$ is quasi-price equilibrium (weak inequality). Definition of preference relation and preference properties are given below.

Definition : Preference relation $\gtrsim$ is a relation $\gtrsim \subset \mathbb{R}_+^l \times \mathbb{R}_+^l$.With properties $x \gtrsim x, \forall x \in \mathbb{R}_+^l$ (reflexivity), $x \gtrsim y, y \gtrsim z \Rightarrow x \gtrsim z$ (transitivity), $\gtrsim$ is a closed set (continuity), $\forall (x \gtrsim y), \exists (y \gtrsim x)$ (completeness) ,given $\gtrsim, \forall\ (x \gg 0)$ the at least good set $\{y: y \gtrsim x\}$ is closed relative to $R^l$(boundary condition),A is convex,if$\{y: y \gtrsim x\}$, is convex set for every $yay + (1-\lambda)x \gtrsim x$,whenever $y \gtrsim x$and $0 < a < 1$, Mas-Colell, A. (1986). This theorem holds if preferences are convex i.e.: The set $A \subset \mathbb{R}^n$ is convex compact and nonempty set if $\lambda x + (1-\lambda)x' \in A, x, x' \in A \wedge \lambda \in [0,1]$.There is a theorem that gives sufficient conditions for the existence of hyperplane separating sets, that is the Separating hyperplane theorem. In geometry hyperplane of an $n$ dimensional vector space $V$ is a subspace of a $n-1$ dimension, or equivalently of codimension 1 in $V$.In geometry, the hyperplane separation theorem is a theorem about disjoint convex sets in n-dimensional Euclidean space.

Theorem 3 Separating hyperplane theorem

Definition of a hyperplane : Hyperplane in $\mathbb{R}^n$ can be described by an equation$\sum_{i=1}^{n} p_i x_i = \alpha$, here vector $p \in \mathbb{R}^n$ is a non-zero price vector, and $\alpha$ is scalar (Simon and Blume 1994, Yu and Phillips 2018)

*Equation 46*

$$H(p, \alpha) = \{x \in \mathbb{R}^n | p \cdot x = \alpha\}$$

Separating hyperplane theorem: Let's suppose that $B \subset \mathbb{R}^n$ is a convex and closed set and $x \notin B, \exists p \in \mathbb{R}^n, p \neq 0, \alpha \in \mathbb{R}, p \cdot x > \alpha, p \cdot y < \alpha, \forall y \in B$.Convex sets $A, B \subset \mathbb{R}^n$ are disjoint $A \cap B = \emptyset, \exists p \in \mathbb{R}^n, p \neq 0, p \cdot x > \alpha, p \cdot y < \alpha, \forall y \in B$.Then there is a hyperplane that separates $A$ and $B$,leaving them on different sides of it. In support of this theorem if $B \subset \mathbb{R}^n$ is convex and $x \notin \text{int}B, \exists p \in \mathbb{R}^n, p \cdot x \geq p \cdot y, p \neq 0$. If $A$ and $B$ are convex, $0 \not\ni A - B, A \cap B = \emptyset$ . Let's say that $S \in R^m$ if $z*$ is a boundary point of set $S, \exists \boldsymbol{p} \neq \boldsymbol{0}, \boldsymbol{z} \in \boldsymbol{S} \rightarrow \boldsymbol{p} \cdot \boldsymbol{z} <= p \cdot \boldsymbol{z}^*$

*Proof lemma 3.1* :if $S$ is a closed and convex set [3], $x \in S$ , and if $b$ is the boundary of this set, then there exists scalar $\alpha \neq 0$ such that: $x \in S \rightarrow \alpha x \leq \alpha b$.Previous theorem ($S, \exists \boldsymbol{p} \neq \boldsymbol{0}, \boldsymbol{z} \in \boldsymbol{S} \rightarrow \boldsymbol{p} \cdot \boldsymbol{z} <= p \cdot z*$, where $p$ is an m-dimensional price vector)holds if $m = 1$,this theorem is also true when $m = n + 1$(Fibich and Gavish 2011)..Here $n + 1$ is the dimension of $S$ production set. Now $z$ is $n + 1$ dimensional vector,$x$ is $n$ dimensional vector and $y$ is one dimensional scalar: $z = (y, x)$, n-dimensional set is:$X(y) \equiv \{x|(y, x) \in S\}$.Now , from these two convex sets $S, X(y)$ follows that:$(\lambda y + (1 - \lambda)y', \lambda x + (1 - \lambda)x') \in S$ , and $x \in X(y*) \rightarrow p_x x \geq p_x x * \blacksquare$.

Asymmetric auctions

There exists literature in the subject of asymmetric auctions namely: Maskin, Riley (2000), Fibich, Gavious (2003),Fibich, Gavish (2011), Güth,et al. (2005),Gayle, Richard (2008), Hubbard, et al. (2013).

Basic setup of the asymmetric auction

There exist set:$\Theta = \{1,2, \dots, N\}$,of types of bidders. And $\forall \theta \in \{1,2, \dots, N\}$and $\exists n(\theta) \geq 1$ , which are bidders of type $\theta$ . Bidders of type $\theta$draw an IPV for the object from CDF$F: [\omega_H, \omega_L] \rightarrow R$ .It is assumed that $F \in C^2((\omega_H, \omega_L))$ and $f \equiv F' > 0$, on $\omega_H$ .The inverse of equilibrium bidding strategy, Maskin and Riley (2000) and Fibich and Gavish (2011) is given as:

*Equation 47*

$$v'_i(b) = \frac{F_i(\beta^{-1}(b))}{f_i(\beta^{-1}(b))} = \left[ \left( \frac{1}{n-1} \sum_{j=1}^{n} \frac{1}{v_j(b) - b} \right) - \frac{1}{v_i(b) - b} \right], i = 1, \dots, n .$$

Solution to the maximization problem given as$\max_b U_i(b;v_i) = (v_i\text{-}b) \prod_{j=1,j\neq 1}^{n} F_j\left(v_j(b)\right)$,i=1,...nis:

*Equation 48*

$$\sum_{j=1, j\neq 1}^{n} \frac{f_j\left(v_j(b)\right) v_j'(b)}{F_j(v_j(b))} - \frac{1}{v_i(b) - b}, i = 1, \dots n .$$

Bidders expected revenue in the FPA asymmetric auction is given as:

*Equation 49*

$$E_i(p, b_i, v_i) = k_i \int_r^{b(\omega_h)} [F_i^{-1}(\ell_i(v)) - v] \cdot \frac{\ell_i'(v)}{\ell_i(v)} \prod_{j=1}^{n} [\ell_j(v)]^{k_j} dv .$$

Bidder maximizes :

[3] Its complement is an open set. Closed set is defined as a set that contains all of its limit points.

*Equation 50*

$$\left(\beta^{-1}(b_1)\right) = arg \max_{u\in(0,\omega_h)} (v-u)\cdot\left[F_i\left(\lambda_i(u)\right)\right]^{k_i-1}\prod_{j\neq 1}\left[F_j\left(\lambda_j(u)\right)\right]^{kj}.$$

$\exists u = \sum_{i=1}^{n} u_i$, where $u_i$ denotes the player of type $i$. Where in previous expressions $\ell_i(v) = F_i(\lambda_i(v))$, and probabilities of winning the reserve price auction are given as:

*Equation 51*

$$p_i(r) = k_i\int_r^{\omega_h}\frac{\ell_i'(v)}{\ell_i(v)}\prod_{j=1}^{n}\left[\ell_j(v)\right]^{k_j}dv\,.$$

Auctioneer expected revenue is given with the following expression:

*Equation 52*

$$E(p,b_i,v_i) = \omega_h - r\prod_{j=1}^{n}\left[F_j(r)\right]^{kj} - \int_r^{b(\omega_h)}\frac{\ell_i'(v)}{\ell_i(v)}\prod_{j=1}^{n}\left[\ell_j(v)\right]^{kj}dv\,.$$

As for comparison in FPA symmetric auction bids and probabilities of winning are given as:

$$\beta(v) = x - \int_{\omega_l}^{\omega_h}\left(\frac{F(y)}{F(x)}\right)^{n-1}dy\ .\Pr\left(v_i \leq \beta^{-1}(b)\right) = F\left(\beta^{-1}(b)\right)^{n-1}$$

Expected revenue in FPA symmetric auction and maximal bid are given as:

$$\mathrm{E}\left(b(v_i)\right) = v_i * \frac{(b_i)^{n-1}}{\left(\frac{n-1}{n-1+a}\right)^{n-1}} \text{or } \mathrm{EE}\left(b(v_i)\right) = v_i * \frac{(b_i)^{n-1}}{\alpha^{n-1}}\ ,\bar{b} = b(1) = 1 - \int_0^1 F^{n-1}(s)\,ds$$

where $a$ is CRRA coefficient, and $\alpha = \left(\frac{n-1}{n-1+a}\right) = \frac{b_i}{v_i}$ this is because $b_i = \alpha * v_i$.The CRRA utility function is given as, Arrow,1965:

*Equation 53*

$$u = \begin{cases}\frac{1}{1-\alpha}c^{1-\alpha}\ if\ \alpha > 0, \alpha \neq 1\\ lnc\ if\ \alpha = 1\end{cases} \text{,when} \alpha = 1\ \Rightarrow \lim_{n\to\infty}\frac{c^{1-\alpha}-1}{1-\alpha} = \ln(c)\,.$$

Elasticity of substitution is $= \frac{1}{\alpha}$, and $MRS = \frac{c_2}{c_1} = \left(\frac{p_1}{p_2}\right)^{\sigma}$,when $\alpha \in [0,1]$ than FPA-bid functions are in form:

*Equation 54*

$$b(v_i) = v - \frac{1}{F^{\frac{n-1}{1-a}}(v)}\int_r^v F^{\frac{n-1}{1-a}}(x)\,dx\text{,or, when } r = 0 \Rightarrow b(v_i) = \frac{n-1}{n-1+a}(v_i).$$

In the previous expression $r$ represents the reserve price. Now, If the coefficient of risk aversion is CARA (Constant absolute risk aversion), i.e. $u(c) = 1 - e^{-ac}$, where $a > 0$, than the bidding function is given as

*Equation 55*

$$b(v_i) = v + \frac{1}{a}\ln\left(1 - \frac{e^{-av}}{F^{n-1}(v)}\int_{e^{ar}}^{e^{av}}\left[F(a^{-1}lnx\right]^{N-1}dx\,.\right.$$

SPA auction NE

In SPA auctions dominant strategy is to bid truthfully i.e.$b(v_i) = v_i$ . The probability to win an auction is given as: $P(v_i) = v_i$, the expected payoff if $b_i$ wins is given as:$\frac{v_i^2}{2}$ . The expected profit in SPA auction and expected revenue,Kunimoto (2008), are given as:

1. $\mathrm{E}\big(b(v_i)\big) = v_i\left(v_i - \frac{v_i}{2}\right) = \frac{v_i^2}{2}$.
2. $Revenue_{(SPA,r=0,a\in\mathbb{R}^+)} = F(y)^n + nF(y)^{n-1}\big(1 - F(y)\big) = nF(y)^{n-1} - (n - 1)F(y)^n$ .

Or in the reserve price auction SPA auction CDF of revenue is given as;

*Equation 56*

$$Revenue_{(SPA,r\in\mathbb{R},a\in\mathbb{R}^+)} = r * r^{n-1} + nF(y)^{n-1} - (n-1)F(y)^n .$$

Uncertain number of bidders

In FPA with uncertain number of bidder equilibrium bid function is given as:

*Equation 57*

$$\beta = \frac{(1-p)\int_o^v xf(x)dx}{p(1-p)F(v)}.$$

Where in previous expression $p_1(1)$ denotes the probability that player 1 believe that he will be only one present at the auction (the only participant). In a symmetric auction with only two bidders we can write $p_1(1) = p, p_1(2) = 1 - p$.CDF is given as:

*Equation 58*

$$F(v) = p(v-b) + (1-p)F\big(\beta^{-1}(b)\big)(v-b).$$

SPA with uncertain number of bidders will still have the same strategy and Nash equilibrium since it will not change the uncertainty of players attendance.

Back to asymmetric auction: weak and strong bidders' equilibrium

Now, let $b_S$ be an equilibrium bid of an strong bidder and $b_w$ is an equilibrium bid of an weak bidder. Than we have following problem to maximize: $\max_b F_w\left(b_w^{-1}(b)\right)(v_S - b)$ .FOC is:

*Equation 59*

$$\frac{f_w\left(b_w^{-1}(b)\right)}{F_w\left(b_w^{-1}(b)\right)} \cdot (b_w^{-1})'(b) - \frac{1}{b_S^{-1}(b) - b}.$$

Strong bidder maximizes

*Equation 60*

$$\frac{f_S\left(b_S^{-1}(b)\right)}{F_S\left(b_S^{-1}(b)\right)} \cdot (b_S^{-1})'(b) = \frac{1}{v_w - b}$$

*Theorem*: Suppose that $F_S(v) \le F_w(v)$, meaning that $F_S$ conditionally first-order stochastically dominates $F_w$.Than when one compares FPA and SPA, both uniformly distributed following applies:

1. $\forall b_S^{-1}(b) = v_S, \because E(b_{FPA}(v)) < E(b_{SPA}(v))\ for\ b_S^{-1}(b) = v_S$
2. $\forall b_w^{-1}(b) \neq v_w, \because E(b_{FPA}(v)) > E(b_{SPA}(v))\ for\ b_w^{-1}(b) \neq v_w$

*Proof:* For purposes of the proof $b_S(v), b_w(v)$ have the same range so a matching function is defined as: $m(v) \equiv b_w^{-1}(b_S(v))$ or as a weak bidder that bids equal to strong bidder in FPA. Since from previous we know that $b_S(v) < b_w(v)$ in FPA, now we know that $m(v) = v$.The strong bidder expected payoff is given as:

*Equation 61*

$$E[\pi(v_i)] = \Pr(b_w(v_w) < b)(v - b)\,.$$

Because $\Pr(v < a) = F(a)$ when distribution of values is uniform. By the envelope theorem Milgrom and Ilya 2002 value function for FPA and SPA (no bid shading)are given as:

*Equation 62*

$$V_S^{FPA}(v) = \int_{\omega_l}^{\omega_h} F_w(m(w))dv\ .V_S^{SPA}(v) = \int_{\omega_l}^{\omega_h} F_w(v)dv\,.$$

Since $m(v) < v$and that $F_w$ is strictly increasing, the strong bidder prefers SPA. For the weak bidders expected payoff for the FPA and SPA are given as:

*Equation 63*

$$V_w^{FPA}(v) = \int_{\omega_l}^{\omega_h} F_S\left(\frac{v}{m}\right) ds\, V_S^{SPA}(v) = \int_{\omega_l}^{\omega_h} F_s(v)dv.$$

Since $m^{-1}(v) > v$, expected payoff is higher for the weak bidder in the FPA.

Numerical example and proof of the Myerson-Satterthwaite theorem

In this part we choose 9 bidder types, there is only one bidder from each type, and these bidders draw their IPVs from for the object of the auction from their CDF $F\colon [\omega_H, \omega_L\ ] \to R$.Ten selected distributions in the following order are, (see,(Johnson, Kemp et al. 2005)) :

*Table 1 Bidders' distributions, boundaries and CDF's*

| **Distributions and boundaries** | **CDF** |
|---|---|
| exponential[0,1], | $F(x) = \frac{1 - \exp\left(-\lambda(x - \omega_L\right)}{1 - \exp\left(-\lambda(\omega_H - \omega_L\right)}$ |
| gamma[0,1], | $F(x) = \frac{\int_0^{\infty} x^{k-1} e^{-x}\, dx}{\Gamma(k)}$ |
| lognormal[0,1], | $F(x) = \frac{\int_a^x \frac{1}{x\sigma\sqrt{2\pi}} exp\left[-\frac{1}{2}\left(\frac{\ln x - \mu}{\sigma}\right)^2\right] dz}{\int_a^b \frac{1}{x\sigma\sqrt{2\pi}} exp\left[-\frac{1}{2}\left(\frac{\ln x - \mu}{\sigma}\right)^2\right] dz}$ |
| Norma[01,] | $F(x) = \int_{-\infty}^{x} \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}$ |

| | |
|---|---|
| standard normal [0,1] | $F(x)=\frac{\Phi\left(\frac{x-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}$ |
| power[0,1] | $F(x)=\frac{\eta}{\alpha+1}\left[(x+a+c)^{\alpha+1}-c^{\alpha+1}\right]$ |
| Reverse power[0,1] | $F(x)=1-\left(\frac{b-x}{b-a}\right)^{\alpha}$ |
| Uniform [0,1] | $F(x)=\frac{x-\omega_L}{\omega_H-\omega_L}$ |
| Weibull [0,1] | $F(x)=\frac{1-exp\left[-\left(\frac{x-\omega_L}{\lambda}\right)^{k}\right]}{1-exp\left[-\left(\frac{\omega_H-\omega_L}{\lambda}\right)^{k}\right]}$ |

On the previous table nine types of bidders' (types of statistical distribution) selected were presented. On the second column Cumulative distribution functions of the selected distribution types were presented. In probability theory and statistics, the cumulative distribution function (CDF) of a real valued random variable $X$,or just the distribution function of $X$,evaluated at $x$, it is probability that:$F_x(x)=P(X\leq x)$,where $P(a\leq x\leq b)=F_x(b)-F_x(a)$, or if it is a continuous random variable CDF of a function can be presented as: $F_x(x)=\int_{-\infty}^{x}f_x(t)dt$ or in the case of absolutely continuous $F_x(x)=\int_{a}^{b}f_x(x)dx$.On the next table results from the Backward shooting method are explained, followed by the small explanation of the computation method used.

*Table 2 Backward shooting method results, reserve price=0.5; end result: Convergence false*

| | b_bar | A | B | B-A | Result | |
|---|---|---|---|---|---|---|
| 0/39 | 0.5 | 0 | 1.00000000 | 1.00E+03 | 3 | Solution not within specified tolerances. |
| 1/39 | 0.75 | 0.5 | 1.00000000 | 5.00E+02 | 3 | Solution not within specified tolerances. |
| 2/39 | 0.875 | 0.75 | 1.00000000 | 2.50E+02 | 3 | Solution not within specified tolerances. |
| 3/39 | 0.9375 | 0.875 | 1.00000000 | 1.25E+02 | 2 | Solution diverges to +Infinity. |
| 4/39 | 0.90625 | 0.875 | 0.9375 | 6.25E+01 | 2 | Solution diverges to +Infinity. |
| 5/39 | 0.890625 | 0.875 | 0.90625 | 3.13E+01 | 1 | Solution diverges to -Infinity. |
| 6/39 | 0.882813 | 0.875 | 0.890625 | 1.56E+01 | 1 | Solution diverges to -Infinity. |
| 7/39 | 0.878906 | 0.875 | 0.8828125 | 7.81E+00 | 1 | Solution diverges to -Infinity. |
| 8/39 | 0.876953 | 0.875 | 0.87890625 | 3.91E+00 | 1 | Solution diverges to -Infinity. |
| 9/39 | 0.875977 | 0.875 | 0.876953125 | 1.95E+00 | 3 | Solution not within specified tolerances. |
| 10/39 | 0.876465 | 0.875977 | 0.876953125 | 9.77E-01 | 3 | Solution not within specified tolerances. |

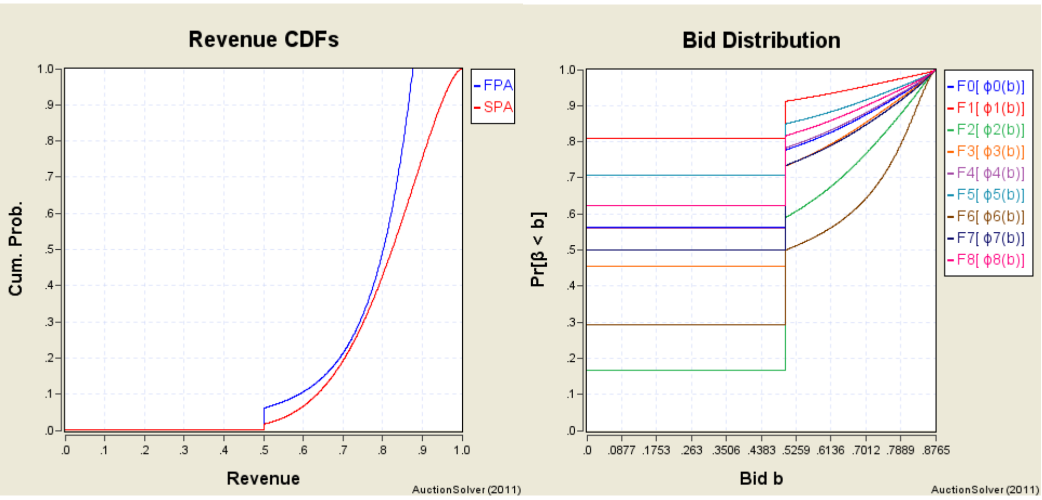
| | | | | | | |
|---|---|---|---|---|---|---|
| 11/39 | 0.876709 | 0.876465 | 0.876953125 | 4.88E-01 | 1 | Solution diverges to -Infinity. |
| 12/39 | 0.876587 | 0.876465 | 0.876708984 | 2.44E-01 | 1 | Solution diverges to -Infinity. |
| 13/39 | 0.876526 | 0.876465 | 0.876586914 | 1.22E-01 | 1 | Solution diverges to -Infinity. |
| 14/39 | 0.876495 | 0.876465 | 0.876525879 | 6.10E-02 | 3 | Solution not within specified tolerances. |
| 15/39 | 0.876511 | 0.876495 | 0.876525879 | 3.05E-02 | 3 | Solution not within specified tolerances. |
| 16/39 | 0.876518 | 0.876511 | 0.876525879 | 1.53E-02 | 3 | Solution not within specified tolerances. |
| 17/39 | 0.876522 | 0.876518 | 0.876525879 | 7.63E-03 | 1 | Solution diverges to -Infinity. |
| 18/39 | 0.87652 | 0.876518 | 0.876522064 | 3.82E-03 | 1 | Solution diverges to -Infinity. |
| 19/39 | 0.876519 | 0.876518 | 0.876520157 | 1.91E-03 | 3 | Solution not within specified tolerances. |
| 20/39 | 0.87652 | 0.876519 | 0.876520157 | 9.54E-04 | 3 | Solution not within specified tolerances. |
| 21/39 | 0.87652 | 0.87652 | 0.876520157 | 4.77E-04 | 3 | Solution not within specified tolerances. |
| 22/39 | 0.87652 | 0.87652 | 0.876520157 | 2.38E-04 | 1 | Solution diverges to -Infinity. |
| 23/39 | 0.87652 | 0.87652 | 0.876520038 | 1.19E-04 | 3 | Solution not within specified tolerances. |
| 24/39 | 0.87652 | 0.87652 | 0.876520038 | 5.96E-05 | 1 | Solution diverges to -Infinity. |
| 25/39 | 0.87652 | 0.87652 | 0.876520008 | 2.98E-05 | 3 | Solution not within specified tolerances. |
| 26/39 | 0.87652 | 0.87652 | 0.876520008 | 1.49E-05 | 1 | Solution diverges to -Infinity. |
| 27/39 | 0.87652 | 0.87652 | 0.87652 | 7.45E-06 | 3 | Solution not within specified tolerances. |
| 28/39 | 0.87652 | 0.87652 | 0.87652 | 3.73E-06 | 3 | Solution not within specified tolerances. |
| 29/39 | 0.87652 | 0.87652 | 0.87652 | 1.86E-06 | 3 | Solution not within specified tolerances. |
| 30/39 | 0.87652 | 0.87652 | 0.87652 | 9.31E-07 | 3 | Solution not within specified tolerances. |
| 31/39 | 0.87652 | 0.87652 | 0.87652 | 4.66E-07 | 1 | Solution diverges to -Infinity. |
| 32/39 | 0.87652 | 0.87652 | 0.87652 | 2.33E-07 | 1 | Solution diverges to -Infinity. |
| 33/39 | 0.87652 | 0.87652 | 0.87652 | 1.16E-07 | 3 | Solution not within specified tolerances. |
| 34/39 | 0.87652 | 0.87652 | 0.87652 | 5.82E-08 | 1 | Solution diverges to -Infinity. |
| 35/39 | 0.87652 | 0.87652 | 0.87652 | 2.91E-08 | 1 | Solution diverges to -Infinity. |
| 36/39 | 0.87652 | 0.87652 | 0.87652 | 1.46E-08 | 3 | Solution not within specified tolerances. |
| 37/39 | 0.87652 | 0.87652 | 0.87652 | 7.28E-09 | 1 | Solution diverges to -Infinity. |
| 38/39 | 0.87652 | 0.87652 | 0.87652 | 3.64E-09 | 3 | Solution not within |

| | | | | | | |
|---|---|---|---|---|---|---|
| 39/39 | 0.87652 | 0.87652 | 0.87652 | 1.82E-09 | 1 | specified tolerances. Solution diverges to -Infinity. |

**Highest Bid:** $\bar{b} = 0.8765200000002551$ **,Shooting Terminated at b = 0.5000000000002981. (b_underbar = 0.5)**

Euler method used in backward shooting solver is described as the simplest Runge-Kutta method, ODE is of the form: $\frac{dy(t)}{dt} = f(t, y(t)),\ y(t_0) = y_0,\ \frac{dy(t)}{dt} \approx \frac{y(t+h)-y(t)}{h},\ y(t+h) \approx y(t) + h\frac{dy}{dt}$ .The iterative solutions is than given as: $y_n + 1 = y_n + hf(t_n, y_n)$ or in previous expressions $x \in (x_0, x_n)$.On the next graph Revenue CDFs and bid distributions are presented under Backward shooting method.

*Figure 1 Revenue CDF and bid distribution with the Backward shooting method*




Next are presented results (inverse bid functions) under Chebychev Approximation and some explanation of the method:

x = x(b) = (b - 0.5) / 0.38906945006425353

Inverse bid functions

1. ϕ0(x) = 0.359684 + 0.742969*T[1] - 0.111041*T[2] - 0.018979*T[3] + 0.045910*T[4] - 0.019714*T[5]
2. ϕ1(x) = 0.400034 + 0.680654*T[1] - 0.106074*T[2] + 0.014502*T[3] + 0.013428*T[4] - 0.004174*T[5]
3. ϕ2(x) = 0.399226 + 0.699775*T[1] - 0.113911*T[2] + 0.003953*T[3] + 0.013566*T[4] - 0.002641*T[5]
4. ϕ3(x) = 0.350580 + 0.760549*T[1] - 0.120452*T[2] - 0.017967*T[3] + 0.046234*T[4] - 0.019976*T[5]
5. ϕ4(x) = 0.352117 + 0.756530*T[1] - 0.121779*T[2] - 0.012382*T[3] + 0.042888*T[4] - 0.018589*T[5]
6. ϕ5(x) = 0.382672 + 0.711359*T[1] - 0.119968*T[2] + 0.015069*T[3] + 0.016516*T[4] - 0.006692*T[5]
7. ϕ6(x) = 0.402056 + 0.702233*T[1] - 0.112901*T[2] + 0.010328*T[3] + 0.008493*T[4] - 0.013409*T[5]

8. ϕ7(x) = 0.345038 + 0.773671*T[1] - 0.136984*T[2] - 0.001167*T[3] + 0.035691*T[4] - 0.016898*T[5]
9. ϕ8(x) = 0.360848 + 0.744004*T[1] - 0.124400*T[2] - 0.000162*T[3] + 0.032328*T[4] - 0.013832*T[5]

High Bid: 0.8890694500642535$\bar{b} \approx 0.89$

About Chebyshev coefficients we pose the following theorem:

*Theorem*:Every function defined on [-1,1], so long as it is at least Lipschitz continuous, has an absolutely and uniformly convergent Chebyshev series:

*equation 64*

$$f(x) \; = a_0 \; + \; a_1 T_1(x) \; + \; a_2 T_2(x) + \dots$$

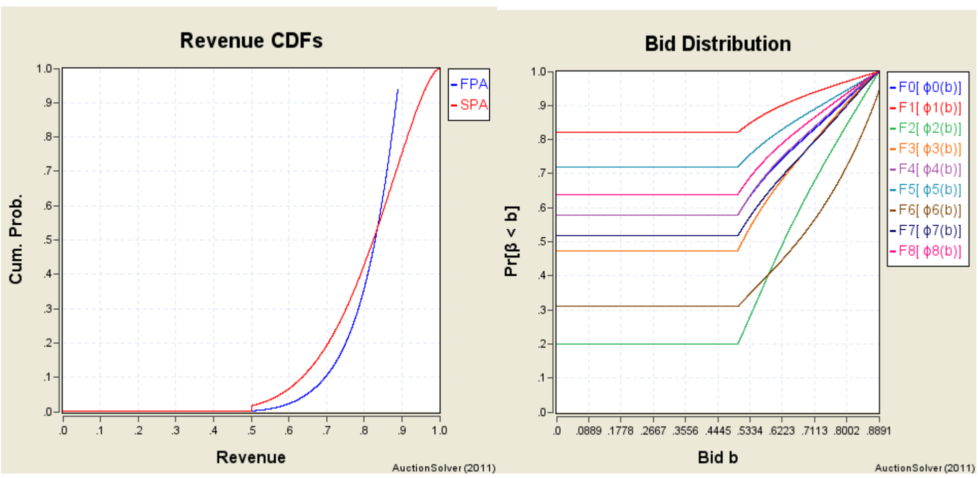
The same holds on an interval [a,b] with appropriately scaled and shifted Chebyshev polynomials.The coefficients of these polynomials for a function $f(x)$ can be obtained by the following integral:

*Equation 65*

$$a_n = \frac{2}{\pi} \int_{-1}^{1} \frac{f(x) T_n(x)}{(1 - x^2)^{\frac{1}{2}}} dx$$

Hubbard ,Paarsch (2009)used Chebyshev polynomials, which are orthogonal polynomials and more stable. Chebyshev nodes can be computed as:$x_t = cos\left[\frac{\pi(t-1)}{T}\right], t = 1, \dots T$.The points $\{v_t\}_t^T = 1$ are found via transformation like this: $v_t = \frac{\bar{b} + \omega_L + (\bar{b} - \omega_L) x_t}{2}$.Graphical depiction of revenue CDFs and bid distribution follows.

*Figure 2 Revenue CDFs and bid distribution with the Chebyshev Approximation*



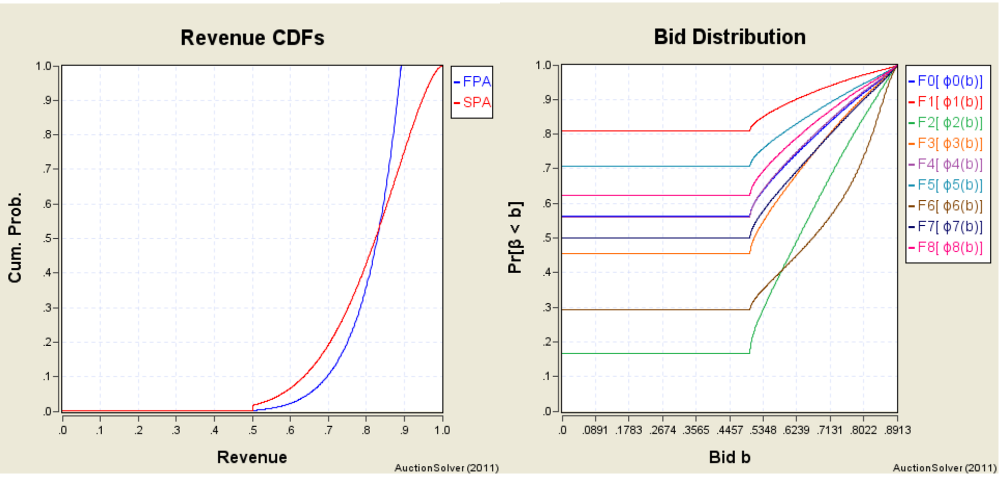

A fixed point is a point that does not change upon of a function (map), system of differential equations etc. (Shashkin 1991) . In the Newton's method the algorithm can be applied iteratively to obtain:$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_{n-1})}$ ,if $\lim_{x_{n+1}\to x^*} \frac{f(x_n)}{f'(x_n)} = x_n$,and $x_n = x^* + \epsilon_n$ ,where $\epsilon_{n+1} = \frac{f''(x^*)}{2\cdot f'(x^*)}\epsilon_n^2$.

*Theorem Fixed point theorem*: if $\exists f(x) \in [a,b]$,then $\exists x \in [a,b]$, and $f(x) - x = 0 \Rightarrow f(x) = x$ , see(Rosenlicht 1968)

Results : convergence true

Next will finish with a depiction on a graph of Revenue CDFs and bid distribution under Fixed point iteration.

*Figure 3 Revenue CDF and bid distribution with the Fixed point iteration*




From the graphs one can conclude that Spence-Mirrleessingle crossing condition holds ,except in the Backward shooting method. The terms SCC (single crossing condition) refers to the requirement that the isoutility curves for agents of different types cross only once, Laffont, Martimort (2002). This can be defined as follows by :

***Assumption* :**

*Equation 66*

$\forall\theta \in \Theta, \forall q \in Q$

$\partial^2_{\theta q} U(q;\theta) \geq 0$

This condition translates in that higher the agent's type $\theta$ , the higher his marginal utility, that is that increase in agent's type means that the agent is willing to trade more, and this is true at all levels of quantities traded $q$. This means that this a requirement of a constant sign on the utility function i.e. second partial derivative $\partial^2_{\theta q}$ of the marginal utility function to be monotone in $\theta$. That means that the agents rent is positive:$r(\theta) \geq 0$ and that $r(\theta) = \max_{\tilde{\theta}} U(q(\tilde{\theta});\theta) - t(\tilde{\theta})$, where $t(\tilde{\theta})$ represents the current transfers. This is important for the incentive compatibility condition to holds and that is to be satisfied allocation rule and the revelation principle that is:

*Allocation rule*:$f: \Theta \to A = Q \times \mathbb{R}; \theta \mapsto f(\theta) = (q(\theta), t(\theta))$

*Revelation principle:*

*Revelation mechanism:* $(M,g):M \rightarrow A$ *and* $h:\Theta \rightarrow M$ *where* $f = g \circ h$ *then it must satisfy:* $\forall \theta, \tilde{\theta} \in \Theta, U(f(\theta),\theta) \geq U(f(\tilde{\theta}),\theta)$ *.So if Spence -Mirrlees condition holds with positive sign then* $q(\cdot), r(\cdot)$ *is IC only and only if:*

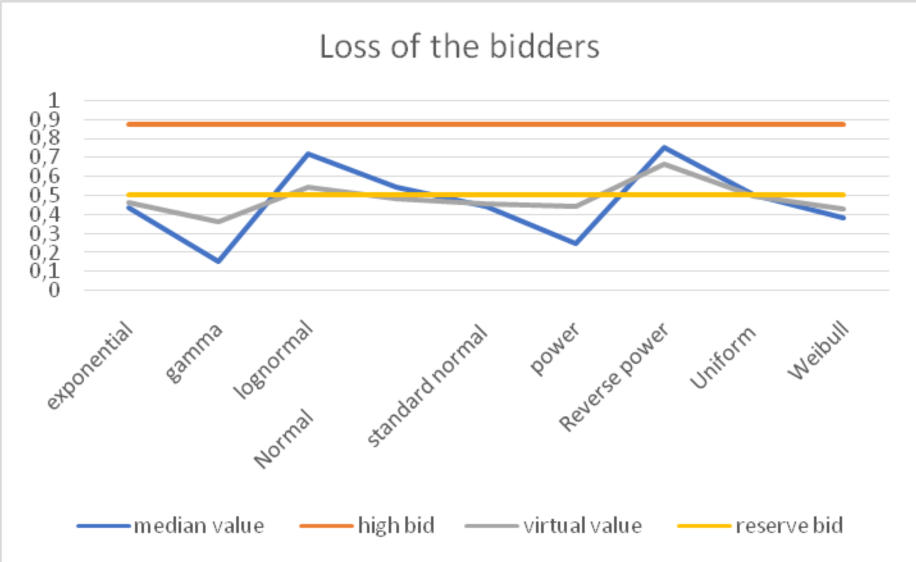
$$\begin{cases} r(q) = r(\underline{\theta}) + \int_{\underline{\theta}}^{\overline{\theta}} \partial_\theta U(q(s);s)\,ds \\ \quad q(\theta)\ ; \Delta q(\theta) > 0 \end{cases}$$

Trade loss is depicted on the following table and graph.

*Table 3 median value, virtual value and loss*

| **Distributions and boundaries** | **Median value** | **Virtual value =0 at x=?** | **Loss** |
|---|---|---|---|
| exponential[0,1], | 0.438140 | 0.467503 | -0.4383800000002551 |
| gamma[0,1], | 0.154211 | 0.363593 | -0.7223090000002551 |
| lognormal[0,1], | 0.713734 | 0.547381 | -0.1627860000002551 |
| Normal[01,] | 0.539199 | 0.486289 | -0.3373210000002551 |
| standard normal [0,1] | 0.441771 | 0.458464 | -0.4347490000002551 |
| power[0,1] | 0.250000 | 0.444444 | -0.6265200000002551 |
| Reverse power[0,1] | 0.750000 | 0.666667 | -0.1265200000002551 |
| Uniform [0,1] | 0.500000 | 0.500000 | -0.3765200000002551 |
| Weibull [0,1] | 0.379885 | 0.432857 | -0.4966350000002551 |

*Figure 4 Median distributions of value of the bidders, virtual valuations and depiction of trade loss*




Virtual valuations of an agent is a function that measures the surplus that can be extracted from that agent. Virtual valuation according to Myerson (1981)is equal to:

*equation 67*

$$r(v) = (b_S^{-1})'(b) - \frac{1 - F(v)}{f(v)}$$

And $\exists n \in [\underline{b}, \overline{b}] \to \mathbb{R}$ so called *revision functions* such thatif another bidder $i$ learned that $(b_S^{-1})'(b)$ was the $j's$ bider valuation estimate for the object, then $i$ would revise his own valuation by $e_j(b_S^{-1})'(b)$ , so that :

*equation 68*

$$v_i((b_S^{-1})'(b)) = (b_S^{-1})'(b) + \sum_{\substack{j \in N \\ j \neq i}} e_j(b_S^{-1})'(b)$$

Where $(b_S^{-1})'(b) = v$,similarly seller raises his bid $s_s((s_S^{-1})'(s)) = (s_S^{-1})'(s) + \sum_{\substack{j \in N \\ j \neq i}} e_j(b_S^{-1})'(b)$, if he learned that $(b_S^{-1})'(b)$ was the vector of estimates held by the bidders. In the case of pure preference uncertainty, we have:

*equation 69*

$$\int_{\underline{b}}^{\overline{b}} e_j v_j f_j(v_j) dv_j = 0$$

Conclusion

Auction in most general terms is a game theoretic mechanism which allocates an object (set of objects) and is composed of set of bidders $N$ , set of objects allocated $O$, a private type space $\mathbb{S}$, and public type space $\Xi$. And where each bidder has type of distributions $\{S_i, \xi_i\} \in \mathbb{S} \times \Xi$,and

$S \times \Xi = \sum_{i=1}^{N} S_i \times \sum_{i=1}^{N} \Xi_i$,which represents thespace of all type profiles, see (Katzwer (2012)). In the First price auction (blind auction),all bidders simultaneously submit sealed bids, so that no bidder knows the bid of any other participant. The highest bidder pays the price they submitted, so this is how this auction differs from SPA auction,Krishna (2009).Effectively First price sealed bid auctions are called tendering for procurement by companies and organizations, eg. government contracts and mining leases. This contrasts to GSPA auctions (or position auctions (sponsored search on Yahoo and Google search engines) generalized by Varian (2006)), where winning bidders pays price offered by the second highest bidder (eg.eBay auction and Google Ads).A mining lease gives the holder the exclusive right to conduct mining operations and sell the minerals specified in the conditions attached to the lease. So how should the bidders' bid in FPSBA? They should bid lower than their valuation (shading price).And while this mechanism leads to high eCPMs (effective cost per thousand impressions,$eCPM = \left(\frac{total\ earnings}{total\ impressions}\right) x\ 1{,}000$.) for the publishers inventory it can lead to unnaturally high prices, as buyers are forced to "guesstimate" how much their competition bid. So, in a game of incomplete information such as auction Bayesian Nash equilibrium is appropriate concept.But problem here is that:The efficient allocation cannotbe achieved by ANY Bayesian Nash Equilibrium in ANY mechanism. This leads in worst case to market failure.FPSBA is not IC (incentive compatible mechanism),since there is no Bayesian-Nash equilibrium in which bidders report their true value. This contrasts basic requirement. Two (IC and ex-post PE (Pareto efficiency)) out of four basic requirements of the Myerson-Satterthwaite theorem (1983) are not satisfied. So,one of the partiesis forced to trade at loss.

**References**


1. Akerlof, G. A. (1970). "The Market for 'Lemons': Quality Uncertainty and the Market Mechanism". Quarterly Journal of Economics. The MIT Press. 84 (3): 488–500.

2. Aliprantis, C. D. and K. C. Border (2013). Infinite Dimensional Analysis: A Hitchhiker's Guide, Springer Berlin Heidelberg.

3. Arrow, K. J. (1965). The theory of risk aversion. Aspects of the Theory of Risk Bearing. Helsinki: YrjoJahnssoninSaatio. Reprinted in: Essays in the Theory of Risk Bearing. Chicago: Markham. 1971. pp. 90–109.
4. Bogachev, V. I. (2007). Measure Theory, Springer Berlin Heidelberg.

5. Fibich, G. and N. Gavish (2011). "Numerical simulations of asymmetric first-price auctions." Games and Economic Behavior**73**(2): 479-495.

6. Gayle, W.-R. and J. F. Richard (2008). Numerical Solutions of Asymmetric, First-Price, Independent Private Values Auctions Computational Economics Volume 32, No. 3, pp 245–278

7. Gibbons,R.(1992)."Game theory for applied economists", Princeton University Press
8. Graham, D., Marshall, R.,(1987). Collusive Bidder Behavior at SingleObject, Second-Price and English Auctions. Journal of Political Economy Vol. 95, No. 6 pp. 1217-1239

9. Güth, W., et al. (2005). Bidding behavior in asymmetric auctions: An experimental study. European Economic Review Vol.49,No.7 pp.1891-1913.

10. Halmos, P. R. (2013). Measure Theory, Springer New York.

11. Harsanyi, John C., (1967/1968). "Games with Incomplete Information Played by Bayesian Players, I-III." Management Science 14 (3): 159-183 (Part I), 14 (5): 320-334 (Part II), 14 (7): 486-502 (Part III).

12. Hewitt, E. and K. Stromberg (2013). Real and Abstract Analysis: A modern treatment of the theory of functions of a real variable, Springer Berlin Heidelberg.

13. Hubbard, T. P. and H. J. Paarsch (2009). "Investigating bid preferences at low-price, sealed-bid auctions with endogenous participation." International Journal of Industrial Organization 27(1): 1-14.
14. Hubbard, T. P., et al. (2013). Using Economic Theory to Guide Numerical Analysis: Solving for Equilibria in Models of Asymmetric First-Price Auctions. Computational Economics Vol.42 , No.2 pp. 241-266.
15. Johnson, N. L., et al. (2005). Univariate Discrete Distributions, Wiley.
16. Kajii, A., Morris, S. (1997). "The Robustness of Equilibria to Incomplete Information". Econometrica. 65 (6): 1283–1309
17. Katzwer, R. M. (2012). Auction Solver User Guide, Princeton University.
18. Khan, M. A. and K. P. Rath (2013). "The Shapley–Folkman theorem and the range of a bounded measure: an elementary and unified treatment." Positivity 17(3): 381-394
19. Klemperer, P.(1999). “Auction Theory: A Guide to the Literature”. Journal of Economic Surveys, Vol. 13, Issue 3
20. Krishna, Vijay. (2009). Auction Theory. Second ed. Academic Press.
21. Kunimoto, T.,(2008), Lecture note on auctions. Department of Economics, McGillUniversity.
22. Laffont, J.J,Martimort, D. (2002). “The Theory of Incentives The Principal-Agent Model”. Princeton, New Jersey: Princeton University Press
23. Mas-Colell A. (1988) Four lectures on the differentiable approach to general equilibrium theory. In: Ambrosetti A., Gori F., Lucchetti R. (eds) Mathematical Economics. Lecture Notes in Mathematics, vol 1330. Springer, Berlin, Heidelberg
24. Maskin, E., Riley,J.,G. (2000). Equilibrium in Sealed High Bid Auctions. Review of Economics Studies. Vol. 67 No.3 pp.439–454
25. Maskin, E., Riley,J.,G. (2000). Equilibrium in Sealed High Bid Auctions. Review of Economics Studies. Vol. 67 No.3 pp.439–454
26. McAfee ,P.(1991).“Efficient Allocation with Continuous Quantities,” Journal of Economic Theory 53, no. 1 ,pp. 51–74.
27. McAfee, P. and McMillan, J. (1987), Auctions and bidding, Journal of Economic Literature Vol. 25, No. 2 ,pp. 699-738
28. McAfee, P. and McMillan, J.(1992), "Bidding Rings", American Economic Review, 82, 579-599.
29. McAfee, R. P., McMillan, J. (1987). Auctions with Entry. Economics Letters Volume 23, Issue 4, pp. 343-347
30. Milgrom, P. 2004. Putting Auction Theory To Work. Cambridge University Press.
31. Milgrom, P., Weber,R.J., (1982). A Theory of Auctions and Competitive Bidding. Econometrica Vol. 50, No. 5 pp. 1089-1122
32. Myerson, R., B. Satterthwaite,M.,A. (1983). Efficient Mechanisms for Bilateral Trading. Journal of Economic Theory. 29 (2): 265–281
33. Myerson, R., B.,(1981). Optimal Auction Design. Mathematics of Operations Research. Vol. 6, No. 1,pp. 58-73
34. Myerson,R.,(1991) “Game Theory: Analysis of Conflict”,Harvard University Press, Cambridge
35. Nachbar,J.(2017).”The Myerson-Satterthwaite theorem”, Washington University
36. Reny,P and McAfee,P.(1992).“Correlated Information and Mechanism Design,” Econometrica 60, no. 2 pp. 395–421
37. Riley, J.,Samuelson, W., (1981). Optimal auctions. The American Economic Review, Vol.71, No.3 pp. 381–392.
38. Robbin, J. W.,Rogers,C.R.,Temple,B. (1987). "On Weak Continuity and the Hodge Decomposition." Transactions of the American Mathematical Society**303**(2): 609-618.

39. Rustichini, A.,Satterthwaite, M. A., Williams, S. R., (1994). "Convergence to Efficiency in a Simple Market with Incomplete Information". Econometrica. 62 (5): 1041–1063.

40. Shashkin, Y. A. (1991). Fixed Points, Universities Press (India) Pvt. Limited.

41. Simon, C. P. and L. Blume (1994). Mathematics for Economists, Norton.
42. Starr, R. M. (1969). "Quasi-Equilibria in Markets with Non-Convex Preferences." Econometrica 37(1): 25-38.
43. Varian, H. R. (2006), "Position auctions". International Journal of Industrial Organization
44. Vickrey, W. (1961). Counterspeculation, Auctions, and Competitive Sealed Tenders. The Journal of Finance Vol.16, No.1 pp. 8–37.
45. Vickrey, W.,(1962). Auction and Bidding Games. In Recent Advances in Game Theory. Princeton, NJ: The Princeton University Conference, pp. 15–27.
46. Vickrey, W.,(1976). Auctions Markets and Optimum Allocations. In Amihud, Y. (ed.), Bidding and Auctioning for Procurement and Allocation: Studies in Game Theory and Mathematical Economics. New York: New York University Press, pp. 13–20
47. Zhou, L. (1993). "A simple proof of the Shapley-Folkman theorem." Economic Theory 3(2): 371-372.