

Union of Mathematicians of Macedonia - ARMAGANKA

**X SEMINAR OF DIFFERENTIAL
EQUATIONS AND ANALYSIS**

and

**2nd CONGRESS OF DIFFERENTIAL
EQUATIONS, MATHEMATICAL ANALYSIS
AND APPLICATIONS**

CODEMA 2022

**Proceedings of the CODEMA 2022
Зборник на трудови од CODEMA 2022**

Skopje, 2023

Уредник: Алекса Малчески
Илустратор: Ана Иванова
Печатница: Ацетони-Прилеп
CD електронска публикација
Тираж: 50

Proceedings of the CODEMA 2022

Зборник трудови од CODEMA 2022

ORGANIZING COMMITTEE:

d-r Aleksa Malcheski, President
d-r Risto Malcheski
d-r Daniel Velinov
d-r Pavel Dimovski
d-r Anevska Katerina
d-r Tomi Dimovski
d-r Zoran Misajleski
d-r Samoil Malcheski
d-r Metodi Glavche
Zlatko Petkovski

HONORARY COMMITTEE:

Boro Piperevski
Borko Ilievski
Lazo Dimov

PROGRAM COMMITTEE:

d-r Risto Malcheski, President
Acad. Stevan Pilipović, Serbia
Acad. Sava Grozdev, Bulgaria
Acad. Gregoris Makrides, Cyprus
Acad. Rosen Nikolaev, Bulgaria
Acad. Prof. Dr. Hari Srivastava
d-r Erdal Karapinar, Turkey
d-r Veselin Nenkov, Bulgaria
d-r Snježana Maksimović, BIH
d-r Ana Vukelić, Croatia
d-r Julije Jakšetić, Croatia
d-r Slobodan Filipovski, Slovenia
d-r Romeo Meštrović, Montenegro
d-r Miomir Andjić, Montenegro
d-r Nenad Vulović, Serbia
dr Aleksandru Mihajlović, Serbia
d-r Jelena Vujaković, Serbia
d-r Alit Ibrahim, N.Macedonia
d-r Daniel Velinov, N.Macedonia
d-r Sanja Atanasova, N.Macedonia
d-r Biljana Zlatanovska, N.Macedonia
d-r Limonka Lazarova, N.Macedonia
d-r Marija Miteva, N.Macedonia

CONTENTS

1. **A GEOGRAPHICALLY WEIGHTED REGRESSION APPROACH IN REGIONAL MODEL FOR REAL ESTATE MASS VALUATION**
Natasha Malijanska, Sanja Atanasova, Gjorgji Gjorgjiev, Igor Peshevski, Daniel Velinov
2. **ПИСА ТЕСТИРАЊЕТО ВО СРЕДНИТЕ СТРУЧНИ УЧИЛИШТА**
Ана Димовска, Томи Димовски
3. **APPLICATION OF MARKOV CHAINS IN BIOLOGY**
Natasha Stojkovikj, Limonka Koceva Lazarova, Aleksandra Stojanova Ilievska
4. **CHAIN CONNECTED SET IN A SPACE**
Zoran Misajleski, Aneta Velkoska, and Emin Durmishi
5. **ABOUT THE T=GEOMETRIC INTRERPRATIONS ON THE BASIC INTERACTIONS AND SOME CONSEQUENCES**
Kostadin Trenchevski
6. **COSINE AND COTANGENT THEOREMS FOR A QUADRILATERAL, TWON NEW FORMULAS FOR ITS AREA AND THEIR APPLICATIONS**
V. Nenkov, St. Stefanov, H. Haimov, A. Velchev
7. **DEFORMED SPHERICAL CURVES**
Marija S. Najdanović, Miroslav D. Maksimović, Ljubica S. Velimirović, Svetozar R. Rančić
8. **ANALYSIS OF STUDENT ACHIEVEMENTS IN TEACHING COMPLEX NUMBERS USING GEOGEBRA SOFTWARE**
Dragana Nedić, Gordana Jotanović, Tijana Paunović, Aleksandar Kršić
9. **SEVERAL LOCI GENERATED BY A MOVING TRIANGLEBETWEEN TWO FIXED CIRCLES**
Prof. Sava Grozdev, Prof. Veselin Nenkov, Assoc. Prof. Tatiana Madjarova
10. **APPLICATION OF SEMIVARIOGRAMS AND KRIGING IN GEOTECHNICAL MODELLING**
Marija Maneva, Igor Peshevski, Milorad Jovanovski, Zoran Misajleski, Daniel Velinov
11. **INVERTIBILITY OF LINEAR COMBINATIONS OF K-POTENT MATRICES**
Marina Tošić, Jelena Vujaković
12. **STATISTICAL ANALYSIS OF REPAIR RATE FOR MAINTENANCE DECISION-MAKING**
Nataša Kontrec, Stefan Panić, Biljana Panić, Mejrima Ljajko
13. **GAMES IN MATHEMATICS INSTRUCTION**
Marina Manić¹, Eugen Ljajko², Marina Tošić
14. **Razvoj na elasticnosta na mislenjeto kaj uchenicite od pochetnoto obrazovanie**
Metodi Glavche, Malcheski Risto
15. **N-TUPLE WEAK ORBITS TENDING TO INFINITY FOR HILBERT SPACE OPERATORS**
Sonja Mančevska, Marija Orovčanec
16. **STEM APPROACH IN TEACHING MATHEMATICS**
Marija Miteva, Limonka Koceva Lazarova, Biljana Zlatanovska, Natasa Stojkovik
17. **АЛГЕБАРСКИ РАЦИОНАЛНИ ИЗРАЗИ ВО ОСНОВНО И СРЕДНО ОБРАЗОВАНИЕ - ПРОБЛЕМИ И НЕДОСТАТОЦИ**
Билјана Златановска¹, Марија Митева¹, Мирјана Коцалева Витанова
18. **CANTOR'S INTERSECTIONS THEOREM IN $(3,1,\nabla)$ -G METRIZABLE SPACES**
Tomi Dimovski, Pavel Dimovski
19. **WORKING WITH MATHEMATICALLY GIFTED STUDENTS AGED 18 TO 19**
Katerina Anevska¹, Valentina Gogovska², Risto Malcheski³,
20. **The Hanh-Banach Theorem for cyclic 2-subspaces generated by 5 and more elements**
Aleksa Malcheski
21. **2-topological vector space**
Aleksa Malcheski
22. **A PARTICULAR SOLUTION TO THE SPECIAL CASE OF A FOURTH-ORDER SHORTENED LORENZ SYSTEM**
Biljana Zlatanovska¹, Boro Piperevski², Mirjana Kocaleva Vitanova¹, Marija Miteva

23. **SOLVING TASKS FROM LINEAR PROGRAMMING USING GEOGEBRA**
Elena Karamazova Gelova ¹, Mirjana Kocaleva Vitanova ², Sonja Mančevska ³

N-TUPLE WEAK ORBITS TENDING TO INFINITY FOR HILBERT SPACE OPERATORS

Sonja Mančevska, Marija Orovčanec

Abstract . In this paper we prove some results on the existence of a dense set of pairs in the direct product of an infinite-dimensional complex Hilbert space with itself such that each pair in this set has an n -tuple weak orbit tending to infinity for a specific countable family of mutually commuting bounded linear operators.

1. INTRODUCTION

For bounded linear operators on Banach spaces the concepts of n -tuple orbits and n -tuple weak orbits are defined as follows. If X is a complex and infinite-dimensional Banach space, $B(X)$ is the algebra of all bounded linear operators on X and $T_1, T_2, \dots, T_n \in B(X)$ are mutually commuting operators, then the n -tuple orbit of the vector $x \in X$ is the set

$$\text{Orb}(\{T_i\}_{i=1}^n, x) = \left\{ T_1^{k_1} T_2^{k_2} \dots T_n^{k_n} x : k_i \geq 0; 1 \leq i \leq n \right\}. \quad (1.1)$$

The n -tuple orbit *tends to infinity* if

$$\lim_{k_i \rightarrow \infty} \|T_1^{k_1} T_2^{k_2} \dots T_n^{k_n} x\| = \infty, \text{ for all } k_j \geq 0, j \neq i, 1 \leq i, j \leq n.$$

For $n=1$, the n -tuple orbit (1.1) reduces to a simple sequence of form

$$\text{Orb}(T, x) = \left\{ T^n x : n = 0, 1, 2, \dots \right\},$$

usually referred as *single orbit* (or simply *orbit*) of the vector $x \in X$ under the operator T . If X^* is the dual space of X , i.e., the space of all bounded linear functionals $x^* : X \rightarrow \mathbb{C}$, and for $x \in X$ and $x^* \in X^*$, $\langle x, x^* \rangle := x^*(x)$, the n -tuple weak orbit of the pair $(x, x^*) \in X \times X^*$ is a set of form

$$\text{Orb}(\{T_i\}_{i=1}^n, x, x^*) = \left\{ \langle T_1^{k_1} T_2^{k_2} \dots T_n^{k_n} x, x^* \rangle : k_i \geq 0; 1 \leq i \leq n \right\}. \quad (1.2)$$

The n -tuple weak orbit *tends to infinity* if

2010 *Mathematics Subject Classification*. Primary: 47A05. Secondary: 47A11, 47A25.

Key words and phrases. Hilbert spaces, weak orbits, n -tuple weak orbits, sequences of operators.

$$\lim_{k_i \rightarrow \infty} \left| \left\langle T_1^{k_1} T_2^{k_2} \dots T_n^{k_n} x, x^* \right\rangle \right| = \infty, \text{ for all } k_j \geq 0, j \neq i, 1 \leq i, j \leq n.$$

For $n=1$, the n -tuple weak orbit (1.2) reduces to a simple scalar sequence of form

$$\text{Orb}(T, x, x^*) = \left\{ \left\langle T^n x, x^* \right\rangle : n = 0, 1, 2, \dots \right\},$$

usually referred as *weak orbit* of the pair $(x, x^*) \in X \times X^*$ under the operator T .

For the case of Hilbert spaces, by the Riesz Theorem for representation of a bounded linear functional on Hilbert spaces (cf. [7, III.6]), given an infinite-dimensional complex Hilbert space H with an inner product $\langle \cdot | \cdot \rangle$, its dual space H^* can be fully identified with the space itself since

$$H^* = \{x \mapsto \langle x | y \rangle, x \in H : y \in H\}.$$

Hence, for a set of mutually commuting operators $T_1, T_2, \dots, T_n \in B(H)$ the n -tuple weak orbits will be the sets of form

$$\text{Orb}(\{T_i\}_{i=1}^n, x, y) = \left\{ \left\langle T_1^{k_1} T_2^{k_2} \dots T_n^{k_n} x | y \right\rangle : k_i \geq 0; 1 \leq i \leq n \right\}, (x, y) \in H \times H.$$

In this paper we will consider only the conditions under which the direct product $H \times H$ contains a dense of pairs (x, y) with n -tuple weak orbits tending to infinity that do not involve any requirements upon specific subsets of the spectra of the operators. For $H \times H$ we will assume that is endowed with the product topology. Given an operator $T \in B(H)$, $\sigma(T)$ and $r(T)$ will denote the spectrum and the spectral radius of the operator T , respectively.

2. PRELIMINARY RESULTS

Theorem 2.1. ([6, Theorem V.39.8]) *Let H and K be Hilbert spaces, $(T_n)_{n \geq 1}$ be a sequence of operators in $B(H, K)$ and $(a_n)_{n \geq 1}$ be sequence of positive numbers with $\sum_{n \geq 1} a_n < \infty$. Then*

- (i) *there are $x \in H$ and $y \in K$ such that and $|\langle T_n x | y \rangle| \geq a_n \|T_n\|$, for all n ;*
- (ii) *there is a dense subset of pairs $(x, y) \in H \times K$ such that $|\langle T_n x | y \rangle| \geq a_n \|T_n\|$, for all n sufficiently large.*

Corollary 2.2. ([6, Corollary V.39.9]) *Let H be Hilbert space and $T \in B(H)$ is such that $\sum_{k=1}^{\infty} \|T^k\|^{-1} < \infty$. Then there exist $x, y \in H$ such that $|\langle T^n x | y \rangle| \rightarrow \infty$. Moreover, the set of such pairs (x, y) is dense in $H \times H$.*

Lemma 2.3. ([6, Lemma V.37.15]) *Let $\varepsilon > 0$ and $(a_n)_{n \geq 1}$ be a sequence of positive numbers satisfying $\sum_{n \geq 1} a_n < \varepsilon$. Then there is a sequence of positive numbers $(b_n)_{n \geq 1}$ such that $b_n \rightarrow \infty$ as $n \rightarrow \infty$ and $\sum_{n \geq 1} a_n b_n < \varepsilon$.*

3. MAIN RESULTS

Let $F = \{1, 2, \dots, N\}$ for some $N \in \mathbb{N}$, $N \geq 2$, or $F = \mathbb{N}$.

Theorem 3.1. *Let H be a Hilbert space, $\{T_i : i \in F\} \subset B(H)$ and $\{(a_{i,j})_{j \geq 1} : i \in F\}$ be a family of sequences of positive numbers such that $\sum_{i \in F, j \geq 1} a_{i,j} < \infty$. Then for any open balls B_1 and B_2 in H there are vectors $x \in B_1$, $y \in B_2$ and $k_0 \in \mathbb{N}$ such that*

$$\left| \langle T_i^k x | y \rangle \right| \geq a_{i,k} \|T_i^k\|, \text{ for all } i \in F \text{ and } k \geq k_0.$$

Proof. Let $T_{i,k} := T_i^k$ ($i \in F$, $k \in \mathbb{N}$), $f : F \times \mathbb{N} \rightarrow \mathbb{N}$ be the bijective mapping defined with

$$f(i, j) = \begin{cases} i + N(j-1), & \text{if } F = \{1, 2, \dots, N\} \\ \frac{(i+j-2)(i+j-1)}{2} + j, & \text{if } F = \mathbb{N} \end{cases},$$

and let $g : \mathbb{N} \rightarrow F \times \mathbb{N}$ denote its inverse mapping. If $(a'_n)_{n \geq 1}$ is a sequence of positive numbers and $(T'_n)_{n \geq 1}$ is a sequence of operators defined with

$$a'_n = a_{g(n)} \text{ and } T'_n = T_{g(n)}, \text{ for all } n \in \mathbb{N},$$

then $\sum_{n \geq 1} a'_n = \sum_{i \in F, j \geq 1} a_{i,j} < \infty$. Hence (by Theorem 2.1. (ii), applied on $(a'_n)_{n \geq 1}$, $(T'_n)_{n \geq 1}$ and $H = K$), if B_1 and B_2 are open balls H , then there are $x \in B_1$, $y \in B_2$ and $n_0 \in \mathbb{N}$ such that

$$\left| \langle T'_n x | y \rangle \right| \geq a'_n \|T'_n\|, \text{ for all } n \geq n_0. \quad (3.1)$$

Since $f : F \times \mathbb{N} \rightarrow \mathbb{N}$ is bijective, there is a unique pair $(i_0, j_0) \in F \times \mathbb{N}$ such that $n_0 = f(i_0, j_0)$. Let

$$k_0 = \begin{cases} j_0 + 1, & \text{if } F = \{1, 2, \dots, N\} \\ i_0 + j_0, & \text{if } F = \mathbb{N} \end{cases}.$$

If $(i, k) \in F \times \mathbb{N}$ is such that $k \geq k_0$, then by the definition of $f : F \times \mathbb{N} \rightarrow \mathbb{N}$ we have:

1. for $F = \{1, 2, \dots, N\}$,

$$\begin{aligned} f(i, k) &= i + N(k-1) \geq N(k_0-1) = Nj_0 = N + N(j_0-1) \\ &\geq i_0 + N(j_0-1) = n_0, \end{aligned}$$

2. for $F = \mathbb{N}$,

$$f(i, k) = \frac{(i+k-2)(i+k-1)}{2} + k \geq \frac{(i_0+j_0-2)(i_0+j_0-1)}{2} + j_0 = n_0.$$

Hence, by (3.1) and the definition of $(a'_n)_{n \geq 1}$ and $(T'_n)_{n \geq 1}$ we obtain

$$\left| \left\langle T_i^k x \middle| y \right\rangle \right| = \left| \left\langle T_{i,k} x \middle| y \right\rangle \right| = \left| \left\langle T_{g(n)} x \middle| y \right\rangle \right| \geq a_{g(n)} \|T_{g(n)}\| = a_{i,k} \|T_i^k\|,$$

for all $i \in F$ and $k \geq k_0$. ■

Theorem 3.2. *If H is Hilbert space and $\{T_i : i \in F\} \subset B(H)$ is a family of operators such that $\sum_{k=1}^{\infty} \|T_i^k\|^{-1} < \infty$, for all $i \in F$, then there is a dense set $D \subset H \times H$ such that the weak orbit $\left(\left\langle T_i^k x \middle| y \right\rangle \right)_{k \geq 0}$ tends to infinity for every pair $(x, y) \in D$ and every $i \in F$. If, in addition, $\{T_i : i \in F\}$ is a family of mutually commuting operators such that the sequence $(T_i^k - T_j^k)_{k \geq 1}$ is norm bounded for all $i, j \in F$, then for every $n \in F$ and $1 < m \leq n$, the m -tuple weak orbit*

$$\left\{ \left\langle T_{i_1}^{k_1} T_{i_2}^{k_2} \dots T_{i_m}^{k_m} x \middle| y \right\rangle : k_i \geq 0; 1 \leq i \leq m \right\},$$

tends to infinity for all $1 \leq i_1 < i_2 < \dots < i_m \leq n$.

Proof. Let B_1 and B_2 be open balls H . For $i \in F$, let $\varepsilon_i > 0$ be such that

$$\varepsilon_i \left(\sum_{k=1}^{\infty} \frac{1}{\|T_i^k\|} \right) < \frac{1}{2^{i+1}},$$

and (by Lemma 2.3) let $(b_{i,k})_{k \geq 1}$ be the sequence of positive numbers such that $b_{i,k} \rightarrow \infty$ as $k \rightarrow \infty$ and

$$\sum_{k=1}^{\infty} \frac{\varepsilon_i b_{i,k}}{\|T_i^k\|} < \frac{1}{2^{i+1}}. \quad (3.2)$$

If $a_{i,k} = \varepsilon_i b_{i,k} \|T_i^k\|^{-1}$, $(i, k) \in F \times \mathbb{N}$, then by (3.2) we have

$$\sum_{i \in F, k \geq 1} a_{i,k} = \sum_{i \in F} \sum_{k=1}^{\infty} \frac{\varepsilon_i b_{i,k}}{\|T_i^k\|} < \sum_{i \in F} \frac{1}{2^{i+1}} < \frac{1}{2}.$$

Hence, by Theorem 3.1, there are $x \in B_1$, $y \in B_2$ and $k_0 \in \mathbb{N}$ such that

$$\left| \left\langle T_i^k x \mid y \right\rangle \right| \geq a_{i,k} \|T_i^k\| = \varepsilon_i b_{i,k} \|T_i^k\|^{-1} \|T_i^k\| = \varepsilon_i b_{i,k}, \text{ for all } i \in F \text{ and } k \geq k_0.$$

Letting $k \rightarrow \infty$, we have

$$\lim_{k \rightarrow \infty} \left| \left\langle T_i^k x \mid y \right\rangle \right| = \infty, \text{ for all } i \in F. \quad (3.3)$$

If, in addition, $\{T_i : i \in F\}$ is a family of mutually commuting operators such that the sequence $(T_i^k - T_j^k)_{k \geq 1}$ is norm bounded for all $i, j \in F$, let $M_{i,j} > 0$ is such that $\|T_i^k - T_j^k\| \leq M_{i,j}$, for all $k \geq 0$, and let $(x, y) \in H \times H$ be a pair satisfying (3.3). We continue by induction.

Let $m = 2$ and $1 \leq i_1 < i_2 \leq n$. By the Cauchy-Bunyakovsky-Schwarz inequality we have

$$\begin{aligned} \left| \left\langle T_{i_1}^{k_1+k_2} x \mid y \right\rangle \right| &\leq \left| \left\langle T_{i_1}^{k_1+k_2} x - T_{i_1}^{k_1} T_{i_2}^{k_2} x \mid y \right\rangle \right| + \left| \left\langle T_{i_1}^{k_1} T_{i_2}^{k_2} x \mid y \right\rangle \right| \\ &= \left| \left\langle T_{i_1}^{k_1} (T_{i_1}^{k_2} - T_{i_2}^{k_2}) x \mid y \right\rangle \right| + \left| \left\langle T_{i_1}^{k_1} T_{i_2}^{k_2} x \mid y \right\rangle \right| \\ &\leq \|T_{i_1}^{k_1} (T_{i_1}^{k_2} - T_{i_2}^{k_2}) x\| \cdot \|y\| + \left| \left\langle T_{i_1}^{k_1} T_{i_2}^{k_2} x \mid y \right\rangle \right| \\ &\leq \|T_{i_1}^{k_1}\| \cdot \|T_{i_1}^{k_2} - T_{i_2}^{k_2}\| \cdot \|x\| \cdot \|y\| + \left| \left\langle T_{i_1}^{k_1} T_{i_2}^{k_2} x \mid y \right\rangle \right| \\ &\leq \|T_{i_1}\|^{k_1} \cdot M_{i_1, i_2} \cdot \|x\| \cdot \|y\| + \left| \left\langle T_{i_1}^{k_1} T_{i_2}^{k_2} x \mid y \right\rangle \right|. \end{aligned}$$

Since $\left| \left\langle T_{i_1}^n x \mid y \right\rangle \right| \rightarrow \infty$ as $n \rightarrow \infty$ (hence $\left| \left\langle T_{i_1}^{k_1+k_2} x \mid y \right\rangle \right| \rightarrow \infty$ as $k_2 \rightarrow \infty$, for all $k_1 \geq 0$), the above inequalities imply that

$$\left| \left\langle T_{i_1}^{k_1} T_{i_2}^{k_2} x \mid y \right\rangle \right| \rightarrow \infty, \text{ as } k_2 \rightarrow \infty, \text{ for all } k_1 \geq 0.$$

Similarly,

$$\begin{aligned} \left| \left\langle T_{i_2}^{k_1+k_2} x \mid y \right\rangle \right| &\leq \left| \left\langle T_{i_2}^{k_1+k_2} x - T_{i_1}^{k_1} T_{i_2}^{k_2} x \mid y \right\rangle \right| + \left| \left\langle T_{i_1}^{k_1} T_{i_2}^{k_2} x \mid y \right\rangle \right| \\ &= \left| \left\langle T_{i_2}^{k_2} (T_{i_2}^{k_1} - T_{i_1}^{k_1}) x \mid y \right\rangle \right| + \left| \left\langle T_{i_1}^{k_1} T_{i_2}^{k_2} x \mid y \right\rangle \right| \\ &\leq \|T_{i_2}\|^{k_2} \cdot M_{i_2, i_1} \cdot \|x\| \cdot \|y\| + \left| \left\langle T_{i_1}^{k_1} T_{i_2}^{k_2} x \mid y \right\rangle \right|, \end{aligned}$$

which implies that

$$\left| \left\langle T_{i_1}^{k_1} T_{i_2}^{k_2} x \mid y \right\rangle \right| \rightarrow \infty, \text{ as } k_1 \rightarrow \infty, \text{ for all } k_2 \geq 0.$$

To complete the proof, it is enough to show the claim is true for $m = n$, under the assumption that the $(n-1)$ -tuple weak orbit

$$\left\{ \left\langle T_{i_1}^{k_1} T_{i_2}^{k_2} \dots T_{i_{n-1}}^{k_{n-1}} x \middle| y \right\rangle : k_j \geq 0; 1 \leq j \leq n-1 \right\},$$

tends to infinity for all $1 \leq i_1 < \dots < i_{n-1} \leq n$. For a fixed $i \in \{1, 2, \dots, n\}$, arbitrary $j \in \{1, 2, \dots, n\} \setminus \{i\}$ and fixed $k_1, k_2, \dots, k_n \geq 0$ we have

$$\begin{aligned} & \left| \left\langle T_1^{k_1} \dots T_{i-1}^{k_{i-1}} T_j^{k_i} T_{i+1}^{k_{i+1}} \dots T_n^{k_n} x \middle| y \right\rangle \right| \\ & \leq \left| \left\langle T_1^{k_1} \dots T_{i-1}^{k_{i-1}} T_j^{k_i} T_{i+1}^{k_{i+1}} \dots T_n^{k_n} x - T_1^{k_1} T_2^{k_2} \dots T_n^{k_n} x \middle| y \right\rangle \right| + \left| \left\langle T_1^{k_1} T_2^{k_2} \dots T_n^{k_n} x \middle| y \right\rangle \right| \\ & = \left| \left\langle T_1^{k_1} \dots T_{i-1}^{k_{i-1}} T_{i+1}^{k_{i+1}} \dots T_n^{k_n} (T_j^{k_i} - T_i^{k_i}) x \middle| y \right\rangle \right| + \left| \left\langle T_1^{k_1} T_2^{k_2} \dots T_n^{k_n} x \middle| y \right\rangle \right| \\ & \leq \left\| T_1^{k_1} \dots T_{i-1}^{k_{i-1}} T_{i+1}^{k_{i+1}} \dots T_n^{k_n} (T_j^{k_i} - T_i^{k_i}) x \right\| \cdot \|y\| + \left| \left\langle T_1^{k_1} T_2^{k_2} \dots T_n^{k_n} x \middle| y \right\rangle \right| \\ & \leq \left(\prod_{\substack{l=1 \\ l \neq i}}^n \|T_l\|^{k_l} \right) \cdot M_{i,j} \cdot \|x\| \cdot \|y\| + \left| \left\langle T_1^{k_1} T_2^{k_2} \dots T_n^{k_n} x \middle| y \right\rangle \right|. \end{aligned}$$

Since $j \in \{1, 2, \dots, n\} \setminus \{i\}$, by the inductive assumption, we have

$$\left| \left\langle T_1^{k_1} \dots T_{i-1}^{k_{i-1}} T_j^{k_i} T_{i+1}^{k_{i+1}} \dots T_n^{k_n} x \middle| y \right\rangle \right| \rightarrow \infty \text{ as } k_i \rightarrow \infty, \text{ for all } k_j \geq 0, j \neq i.$$

This, together with the above inequalities implies that

$$\left| \left\langle T_1^{k_1} T_2^{k_2} \dots T_n^{k_n} x \middle| y \right\rangle \right| \rightarrow \infty \text{ as } k_i \rightarrow \infty, \text{ for all } k_j \geq 0, j \neq i,$$

which completes the proof. ■

Corollary 3.3. *If H is a Hilbert space and $\{T_i : i \in F\} \subset B(H)$ is a family of operators such that $r(T_i) > 1$, for all $i \in F$, then there is a dense set $D \subset H \times H$ such that the weak orbit $\left(\left\langle T_i^k x \middle| y \right\rangle \right)_{k \geq 0}$ tends to infinity for every pair $(x, y) \in D$ and every $i \in F$. If, in addition, $\{T_i : i \in F\}$ is a family of mutually commuting operators such that the sequence $(T_i^k - T_j^k)_{k \geq 1}$ is norm bounded for all $i, j \in F$, then for every $n \in F$, every $1 < m \leq n$ the m -tuple weak orbit*

$$\left\{ \left\langle T_{i_1}^{k_1} T_{i_2}^{k_2} \dots T_{i_m}^{k_m} x \middle| y \right\rangle : k_i \geq 0; 1 \leq i \leq m \right\},$$

tends to infinity for all $1 \leq i_1 < i_2 < \dots < i_m \leq n$.

Proof. If $T \in B(H)$ has a spectral radius $r(T) > 1$, then $\sum_{k=1}^{\infty} \|T^k\|^{-1} < \infty$. Namely, if $r(T) > 1$, then there is $\lambda \in \sigma(T)$ such that $1 < |\lambda|$. By the Spectral Mapping Theorem, $\lambda^n \in \sigma(T^n)$ for every $n \in \mathbb{N}$. Hence $|\lambda|^n \leq r(T^n) \leq \|T^n\|$ and

$$\sum_{n=1}^{\infty} \frac{1}{\|T^n\|} \leq \sum_{n=1}^{\infty} \frac{1}{|\lambda|^n} < \infty.$$

Now the conclusion follows from Theorem 3.2. ■

4. REMARKS ON N-TUPLE ORBITS TENDING TO INFINITY

By the Cauchy-Bunyakovsky-Schwarz inequality we have

$$\left| \left\langle T_1^{k_1} T_2^{k_2} \dots T_n^{k_n} x \middle| y \right\rangle \right| \leq \|T_1^{k_1} T_2^{k_2} \dots T_n^{k_n} x\| \cdot \|y\|,$$

for all $(x, y) \in H \times H$, $k_j \geq 0$ and $1 \leq j \leq n$. These inequalities clearly imply that the n -tuple orbit $\text{Orb}(\{T_i\}_{i=1}^n, x)$ tends to infinity whenever there is $y \in H$ such that the n -tuple weak orbit $\left\{ \left\langle T_1^{k_1} T_2^{k_2} \dots T_n^{k_n} x \middle| y \right\rangle : k_i \geq 0; 1 \leq i \leq n \right\}$ tends to infinity. Hence, from the results in the previous section we can derive the following results for n -tuple orbits tending to infinity.

Theorem 4.1. *If H is Hilbert space and $\{T_i : i \in F\} \subset B(H)$ is a family of operators such that $\sum_{k=1}^{\infty} \|T_i^k\|^{-1} < \infty$ for all $i \in F$, then there is a dense set $D \subset H$ such that the orbit $\text{Orb}(T_i, x)$ tends to infinity for every $x \in D$ and every $i \in F$. If, in addition, $\{T_i : i \in F\}$ is a family of mutually commuting operators such that the sequence $(T_i^k - T_j^k)_{k \geq 1}$ is norm bounded for all $i, j \in F$, then for every $n \in F$, every $1 < m \leq n$, the m -tuple orbit $\text{Orb}(\{T_{i_j}\}_{j=1}^m, x)$ tends to infinity for all $1 \leq i_1 < i_2 < \dots < i_m \leq n$.*

Corollary 4.2. *If H is Hilbert space and $\{T_i : i \in F\} \subset B(H)$ is a family of operators such that $r(T_i) > 1$ for all $i \in F$, then there is a dense set $D \subset H$ such that the orbit $\text{Orb}(T_i, x)$ tends to infinity for every $x \in D$ and every $i \in F$. If, in addition, $\{T_i : i \in F\}$ is a family of mutually commuting operators such that the sequence $(T_i^k - T_j^k)_{k \geq 1}$ is norm bounded for all $i, j \in F$, then for every $n \in F$, every $1 < m \leq n$, the m -tuple orbit $\text{Orb}(\{T_{i_j}\}_{j=1}^m, x)$ tends to infinity for all $1 \leq i_1 < i_2 < \dots < i_m \leq n$.*

COMPETING INTERESTS

Authors have declared that no competing interests exist.

References

- [1] S. Mančevska, M. Orovčanec, *Few results on weak orbits under sequences of operators*, Proceedings of the Fourth International Scientific Conference – FMNS2011, Vol.2, (2011) p.33-40.
- [2] S. Mančevska, M. Orovčanec, *N-Tuple Orbits tending to infinity*, E-Proceedings of the First Congress of Differential Equations, Mathematical Analysis and Applications CODEMA 2020, 24–31.
- [3] S. Mančevska, *N-Tuple Orbits tending to infinity for Hilbert space operators*, Mat. Bilten 45. No. 2 (2021), 143–148.
- [4] S. Mančevska, M. Orovčanec, *Orbits tending to infinity under sequences of operators on Hilbert spaces*, Filomat 21:2 (2007), 163–173.
- [5] V. Müller, J. Vršovský, *Orbits of linear operators tending to infinity*, Rocky Mountain J. Math., Vol. 39 No. 1(2009), 219–230.
- [6] V. Müller, *Spectral theory of linear operators and spectral systems in Banach algebras*, (2nd ed.), Operator Theory: Advances and Applications Vol. 139, Birkhäuser Verlag AG, Basel - Boston - Berlin, 2007.
- [7] K. Yoshida, *Functional Analysis*, 5th ed. Springer-Verlag, Berlin, Heidelberg, 1974.

Faculty of Information and Communication Technologies, University "St.
Kliment Ohridski", Bitola, Republic of North Macedonia
E-mail address: sonja.manchevska@uklo.edu.mk

Institute of Mathematics, Faculty of Natural Sciences and Mathematics,
University of Ss. Cyril and Methodius, Skopje, Republic of North Macedonia
E-mail address: marijaor@iunona.pmf.ukim.edu.mk