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Some Conditions on Total Boundness $(3,2,\rho)$-Symmetric Spaces

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Abstract- **In this paper some necessary and sufficient conditions under which $(3,2,\rho)$ −symmetric spaces are totally bounded are given. It will be shown that a $(3,2)-N$ −symmetrizable topological space (M,τ) is totally bounded $(3,2)$ −symmetric space if and only if it is a regular second-countable space.**

Keywords-**totally bounded $(3,2,\rho)$ −symmetric, totally bounbed $(3,2,\rho)$ −symmetric spaces, totally bounded $(3,2)-N$ −symmetrizable spaces**

I. Introduction

This paper gives a specific meaning to a well known general Theorem: A metrizable space is metrizable by a totally bounded metric if and only if it is a separable space, refer to [12]. Proving this Theorem for classes of $(3,2,\rho)-$ symmetric spaces gives a specific characterization on these topologies, namely the equivalence of the class of separable spaces to the class of metriziable spaces by a totally bounded metric. We are interested in $(3,2,\rho)$ −symmetric and $(3,2)-N$ −symmetrizable spaces because many important topological spaces used in various branches of mathematics are metrizable with a special types of metrics such as $(3,2,\rho)$ −symmetric.

For the notion of $(3,2,\rho)-$ symmetric space we refer to [4]. The necessary definitions are already given there, however in this paper we will only mention the ones that are of interest for understanding the properties of boundness. For better understandings we give couple of examples for totally bounded $(3,2)-$ symmetric spaces.

We then discuss some classes of totally bounded $(3,2)-$ symmetric spaces, and we will show that the class of all spaces metrizable by a totally bounded $(3,2)-$ symmetric coincides with the class of all separable $(3,2)$ −metrizable spaces. As we shall see, the last fact yields an internal characterization of the class of all spaces $(3,2)$ −metrizable by a totally bounded $(3,2)-$ symmetric. This is a very important property for a large class of topological spaces.

II. Definitions and some properties of $(3,2,\rho)-N-$ symmetric spaces

In this part we state the notions (defined in [4]) used letter. Let M be a nonempty set, and let $d: M^3 \to \mathrm{R}_0^+ = [0,\infty)$. We state five conditions for such a map.

(M0) $\boldsymbol{d(x,x,x)=0}$, forany $x \in M$;

(P) $\boldsymbol{d(x,y,z)=d(y,z,x)=d(z,x,y)}$, for any $x,y,z \in M$;

(M1) $d(x,y,z) \le d(x,y,a)+d(x,a,z)+d(a,y,z)$, for any $x,y,z,a \in M$;

(M2) $d(x,y,z) \le d(x,a,b)+d(a,y,b)+d(a,b,z)$, for any $x,y,z,a,b \in M$;

(Ms) $\boldsymbol{d(x,x,y)=d(x,y,y)}$, forany $x,y \in M$.

For a map d as above let $\rho=\{(x,y,z)|(x,y,z) \in M^3, d(x,y,z)=0\}$. The set ρ is a $(3,j)$ −equivalence on M, as defined and discussed in [10] and [4]. The set $\Delta=\{(x,x,x)|x \in M\}$ is a $(3,j)$ −equivalence on M, $j=1,2$, and the set $\nabla=\{(x,x,y)|x,y \in M\}$ is a $(3,1)$ −equivalence, but it is not a $(3,2)$ −equivalenceon M. The condition **(M0)** implies that $\Delta \subseteq \rho$.

Definition 2.1. Let $d: M^3 \to \mathrm{R}_0^+$ and ρ be as above. If d satisfies **(M0)**, **(P)** and **(Mj)**, $j \in \{1,2\}$, we say that d is a $(3,j,\rho)$ **−metric** on M. If d is a $(3,j,\Delta)$ −metric on M, we say that d is a $(3,j)$ **−metric** on M. If d is a $(3,j,\rho)$ **−metric** and satisfies **(Ms)**, we say that d is a $(3,j,\rho)$ **−symmetric** on M, and if d is a $(3,\rho)$ **−metric** and satisfies **(Ms)**, we say that d is a $(3,\rho)$ **−symmetric** on M.

Remark 2.1. Any $(3,j,\rho)$ −metric d on M induces a map $D_d: M^2 \to \mathrm{R}_0^+$ defined by:

$$D_d(x,y)=d(x,x,y).$$

It is easy to check the following facts.

a) For any$(3,j,\rho)$ −metric d, $D_d(x,x)=0$. (D_d is called a **distance** in [16] and a **pseudo o-metric** in [18].)

b) For any $(3,j)$ −metric d, $D_d(x,y)=0$ if and only if $x=y$. (D_d is called an ***o*-metric** in [18].)

c) For any $(3,j)$ −symmetric d, $D_d(x,y)=D_d(y,x)$. (D_d is called a **symmetric** in [18].)

d) For any $(3,2,\rho)$ −metric d, $D_d(x,y) \le 2D_d(z,x)+D_d(z,y)$, and $D_d(x,y) \le 2D_d(y,x)$.

e) For any $(3,2)$ −symmetric d,

$$D_d(x,y)=D_d(y,x) \le 3\big(D_d(x,z)+D_d(z,y)\big)/2.$$

(In the literatureD_d is called a **quasimetric, an earmetrics** or an **inframetrics**.)

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Let d be a $(3,2,\rho)$ −metric on $M, x, y \in M$and $\varepsilon > 0$. As in [4], we consider the following ε-ball, as subsets of M.

$\boldsymbol{B(x,y,\varepsilon) = \{z | z \in M, d(x,y,z) < \varepsilon\} - \varepsilon}$ −ball with center at (x,y) and radius ε.

Remark 2.2. For $x = y$, $B(x,x,\varepsilon) = B(x,y,\varepsilon) = \{z | z \in M, d(x,x,z) < \varepsilon\}$, and $a \in B(a,a,\varepsilon)$, but, it is possible for some $x \neq a$ to have $a \notin B(a,x,\varepsilon)$.

Among the others, a $(3,2,\rho)$ −metric$d$ on M induces the following two topologies as in [4]:

1) $\boldsymbol{\tau(N,d)}$ −the topology defined by: $U \in \tau(N,d)$ iff $\forall x \in U$, $\exists \varepsilon > 0$ such that $B(x,x,\varepsilon) \subseteq U$;
2) $\tau(D,d)$ −the topology generated by all the ε −balls $B(x,x,\varepsilon)$.

In [4] we proved that $\tau(N,d) \subseteq \tau(D,d)$ for any $(3,2,\rho)$ −metric d and in [8] we proved that $\tau(N,d) = \tau(D,d)$ for any $(3,2,\rho)$ −symmetric d.

Definition 2.2. We say that a topological space (M,τ) is $(3,2,\rho) - N$ −symmetrizable via a $(3,2,\rho)$ −symmetric d on M, if $\tau = \tau(N,d)$.

With next definitions we defined the notions for totally bounded (3,2) −symmetric space and (3,2) − N −symmetrizable totally boundedspace.

Definition 2.3. Let (M, D_d) be a (3,2) − symmetric space and A a subset of M. We say that A is **$\boldsymbol{\varepsilon}$ −dense** in (M, D_d) if for every $x \in M$ there exists an $x' \in A$ such that $D_d(x, x') < \varepsilon$.

Definition 2.4. A (3,2) − symmetric space (M, D_d) is **totally bounded** if for every $\varepsilon > 0$ there exists a finite set $A \subset M$ which is **$\boldsymbol{\varepsilon}$** −dense in (M, D_d).

Definition 2.5. A (3,2) − symmetric D_d on a set M is **totally bounded** if the space (M, D_d) is totally bounded. A topological space (M,τ) is $(3,2) - N$ **−symmetrizable by a totally bounded** if there exists a totally bounded (3,2) − symmetric D_d.

Next we give the promised examples.

Example 1. a) The (3,2) − symmetrizable Hilbert space H, constructed in [8] is totally bounded (3,2) − symmetric space.

b) Every closed interval $I = [a,b] \subset \mathbb{R}$ is totally bounded (3,2) − symmetric space. Indeed, for every $\varepsilon > 0$ the set $I \cap \left\{\frac{i}{k} \middle| k \in \mathbb{N}, i = 0 \pm 1, \pm 2, \ldots\right\}$, where k satisfying $\frac{1}{k} < \varepsilon$, is finite and ε −dense in I. Also every open interval is totally bounded (3,2) −symmetric space. Thus, since the real line is homeomorphic to the interval $(-1,1)$ we can see that a space homeomorphic to a totally bounded (3,2) − symmetric space need not be totally bounded (3,2) − symmetric space, it is, however, metrizable by a totally bounded (3,2) − symmetric. We can readily verify that a space isometric to a totally bounded (3,2) −symmetric space is totally bounded

c) Let $\{M_s, D_{d_s}\}_{s\in S}$ is a family of non-empty (3,2) −symmetric spaces such that the (3,2) −symmetric (3,2) −symmetric spaces such that the (3,2) −symmetric D_{d_s} is bounded by 1 for every $s \in S$, then the sum $\bigoplus_{s\in S} M_s$ with the (3,2) −symmetric D_d defined with

$$D_d(x,y) = \begin{cases} D_{d_s}(x,y), if\ x,y \in M_s\ f\ orsomes \in S, \\ 1,\ \ or\ else \end{cases},$$

is totally bounded (3,2) −symmetric if and only if all (3,2) −symmetric spaces (M_s, D_{d_s}) are totally bounded and $|S| < \aleph_0$.

III. Some properties for totally bounded $(3,2) - N$ −symmetrizable space

Here we will prove the main statement that gives an internal characterization of the class of all spaces (3,2) −metrizable by a totally bounded (3,2) − symmetric.

We begin by proving some necessary propositions.

Proposition 3.1. If (M, D_d) is totally bounded (3,2) −symmetric space, then for every subset A of M the (3,2) −symmetric spaces (A, D_d) is totally bounded (3,2) −symmetric space.

If (M, D_d) is an arbitrary (3,2) −symmetric spaces and for subset A of M the space (A, D_d) is totally bounded (3,2) −symmetric space, then the space $(\overline{A}, D_d)$ also is totally bounded (3,2) −symmetric spaces.

Proof. Taka an $\varepsilon > 0$ and a finite set $B = \{x_1, x_2, \ldots, x_k\}$ which is $\frac{\varepsilon}{3}$ −dense in (M, D_d). Let $\{x_{m_1}, x_{m_2}, \ldots, x_{m_l}\}$ be a subset of B consisting of all points whose distance from A is less than $\frac{\varepsilon}{3}$ and let $x_1', x_2', \ldots, x_l'$ be arbitrary points of A satisfying

$$D_d\left(x_j', x_{m_j}\right) < \frac{\varepsilon}{3} \ \text{ for } j = 1,2,\ldots,l. \qquad (1)$$

We shall show that the set $B' = \{x_1', x_2', \ldots, x_l'\}$ is ε −dense in A. Let x be a point of A, by the definition of B there exists an $i \leq k$ such that

$$D_d(x, x_i) < \frac{\varepsilon}{3}. \qquad (2)$$

Hence $x_i = x_{m_j}$ for some $j \leq l$ and, by (1) and (2) we have

$$D_d(x, x_j') \leq \frac{3\left(D_d(x,x_i) + D_d(x_i, x_j')\right)}{2} < \varepsilon.$$

The second part of the proposition follows from the easily observed fact that any set which is $\frac{\varepsilon}{3}$ −dense in (A, D_d) is ε −dense in $(\overline{A}, D_d)$. ■

Proposition 3.2. Let $\{(M_i, D_{d_i})\}_{i=1}^{\infty}$ be a family of non-empty (3,2) −symmetric spaces such that the (3,2) −symmetric D_{d_i} is bounded by 1 for $i = 1,2, \ldots$ The Cartesian product $\prod_{i=1}^{\infty} M_i$ with the (3,2) −symmetric D_d defined with

$$D_d(x,y) = \sum_{i=1}^{\infty} \frac{1}{2^i} D_{d_i}(x_i, y_i),$$

is totally bounded (3,2) −symmetric if and only if all (3,2) −symmetric spaces (M_i, D_{d_i}) are totally bounded.

Proof. Assume that the (3,2) −symmetricspace $(\prod_{i=1}^{\infty} M_i, D_d)$ is totally bounded. The subspace $M_j^* = \prod_{i=1}^{\infty} A_i$ of $\prod_{i=1}^{\infty} M_i$, where $A_j = M_j$ and A_i is an arbitrarily chosen one-point subset of M_i for $i \neq j$, is totally bounded by virtue of Proposition 3.1. One can easily verify that if a set A is $\frac{\varepsilon}{2^j}$ −dense in (M_j^*, D_d), then the set $p_j(A)$ is ε −dense in

$\left(M_j, D_{d_j}\right)$, so that $(3,2)$ −symmetricspace $\left(M_j, D_{d_j}\right)$ is totally bounded.

Assume now that all $(3,2)$ −symmetricspaces $\left(M_j, D_{d_j}\right)$ are totally bounded. Take an $\varepsilon > 0$ and a natural number k such that $\frac{1}{2^k} < \frac{\varepsilon}{3}$. For every $i \leq k$ choose a finite set $\{x_1^i, x_2^i, \dots, x_{m(i)}^i\}$ which is $\frac{\varepsilon}{3}$ −dense in M_i and for every $i > k$ choose an arbitrary point $x_0^i \in M_i$.

The set $A \subset \prod_{i=1}^{\infty} M_i$ consisting of all points of the form

$$y = \{x_{j_1}^1, x_{j_2}^2, \dots, x_{j_k}^k, x_0^{k+1}, x_0^{k+2}, \dots\},$$
$$\text{where } 1 \leq j_i \leq m(i) \text{ for } i \leq k, \qquad (3)$$

is finite. To conclude the proof it suffices to show that A is ε −dense in the $(3,2)$ −symmetricspace $(\prod_{i=1}^{\infty} M_i, D_d)$.

Let $x = \{x_i\}$ be an arbitrary point of $\prod_{i=1}^{\infty} M_i$. For every $i \leq k$ there exists a $j_i \leq m(i)$ such that $D_{d_i}\left(x_i, x_{j_i}^i\right) < \frac{\varepsilon}{3}$ and for the point y defined in (3) we have

$$D(x,y) = \sum_{i=1}^{k} \frac{1}{2^i} D_{d_i}\left(x_i, x_{j_i}^i\right) + \sum_{i=k+1}^{\infty} \frac{1}{2^i} D_{d_i}\left(x_i, x_0^i\right)$$
$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} < \varepsilon,$$

so that the set aA is ε −dense in $(\prod_{i=1}^{\infty} M_i, D_d)$. ■

Corollary 3.1. The Hilbert cube $I^{\aleph_0}$ with the $(3,2)$ −symmetric D_d defined by letting

$$D_d(x,y) = \sum_{i=1}^{\infty} \frac{1}{2^i} |x_i - y_i|,$$
$$\text{where } x = \{x_i\} \text{ and } y = \{y_i\},$$

is a totally bounded space. ■

Proposition 3.3. A $(3,2) - N$ −symmetrizable space is metrizable by a totally bounded $(3,2)$ −symmetric D_d if and only if is a separable space.

Proof. From the statement that: the Hilbert cube $I^{\aleph_0}$ is universal for all compact metrizable spaces and for all separable metrizable spaces, from corollary 3.1 and proposition 3.1 follows the sufficiency of separability.

To prove that separability also is a necessary condition, it is enough to observe that if D_d is a totally bounded $(3,2)$ −symmetric on a space M and $A_i \subset M$ is a finite set which is $\frac{1}{i}$ −dense in (M, D_d), then the union $A = \bigcup_{i=1}^{\infty} A_i$ is a countable dense subset of M. ■

From the above statements and the proof in [6] that $(3,2) - N$ −symmetrizable space is regular, follows.

Corollary 3.2. A topological space (M, τ) is $(3,2) - N$ −symmetrizable by a totally bounded $(3,2)$ −symmetric D_d if and only if is a regular second-countable space. ■

IV. Conclusion

We have given characterization of the class of all spaces $(3,2,\rho)$ −metrizable by a totally bounded $(3,2)$ − symmetric. By showing that $(3,2) - N$ −symmetrizable space is metrizable by a totally bounded $(3,2)$ −symmetric D_d if and only if is a separable space, we proved the equivalence of the class of separable spaces to the class of totally bounded $(3,2)$ − symmetric spaces. By this we have given a specific meaning and importance of one general theorem, that is of great importance for a large classes of topological spaces that are widely used in applications.

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