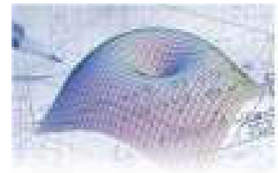

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N-Tuple Weak Orbits Tending to Infinity for Banach Space Operators

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Abstract: In this paper we prove some results on the existence of a dense set of pairs in the product of an infinite-dimensional complex Banach space with its dual space such that each pair of this set has an $n-$tuple weak orbit tending to infinity for specific countable family of mutually commuting bounded linear operators.

Keywords: Banach spaces, weak orbits, n-tuple weak orbits, sequences of operators.

1. Introduction

Let X be an infinite-dimensional Banach space over the field of complex numbers $\mathbb{C}$, $B(X)$ the algebra of all bounded linear operators on X and X^* its dual space, i.e., the space of all bounded linear functionals $x^*: X \to \mathbb{C}$. For the direct product $X \times X^*$ we assume that is endowed with the product topology. As usually, if $x \in X$ and $x^* \in X^*$, we will denote $\langle x, x^* \rangle := x^*(x)$. If $T \in B(X)$, then $\sigma(T)$ and $r(T)$ will denote the spectrum and the spectral radius of the operator T, respectively.

If $T_1, T_2, \dots, T_n \in B(X)$ are mutually commuting operators, $n-$*tuple orbit* of the vector $x \in X$ (or, the *orbit of* x under the $n-$tuple $\mathbf{T} = (T_1, T_2, \dots, T_n)$) is the set

$$\mathrm{Orb}(\{T_i\}_{i=1}^{n}, x) = \{T_1^{k_1} T_2^{k_2} \dots T_n^{k_n} x : k_i \geq 0; 1 \leq i \leq n\}. \tag{1.1}$$

The n −tuple orbit *tends to infinity* if

$$\lim_{k_i \to \infty} \|T_1^{k_1} T_2^{k_2} \dots T_n^{k_n} x\| = \infty, \text{ for all } k_j \geq 0, j \neq i, 1 \leq i, j \leq n.$$

For $n = 1$, the n −tuple orbit (1.1) reduces to a simple sequence of form

$$\mathrm{Orb}(T, x) = \{T^n x : n = 0,1,2, \dots\},$$

usually referred as *single orbit* (or, simply *orbit*) of the vector $x \in X$ under the operator T.

The n *−tuple weak orbit* of the pair $(x, x^*) \in X \times X^*$ (or, the *weak orbit of the pair* (x, x^*) under the n −tuple $\mathbf{T} = (T_1, T_2, \dots, T_n)$) is the set

$$\mathrm{Orb}(\{T_i\}_{i=1}^{n}, x, x^*) = \{\langle T_1^{k_1} T_2^{k_2} \dots T_n^{k_n} x, x^* \rangle : k_i \geq 0; 1 \leq i \leq n\}. \tag{1.2}$$

The n −tuple weak orbit *tends to infinity* if

$$\lim_{k_i \to \infty} |\langle T_1^{k_1} T_2^{k_2} \dots T_n^{k_n} x, x^* \rangle| = \infty, \text{ for all } k_j \geq 0, j \neq i, 1 \leq i, j \leq n.$$

For $n = 1$, the n −tuple weak orbit (1.2) reduces to a sequence of form

$$\mathrm{Orb}(T, x, x^*) = \{\langle T^n x, x^* \rangle : n = 0,1,2, \dots\},$$

usually referred as *weak orbit* of the pair $(x, x^*) \in X \times X^*$ under the operator T.

In [5] we gave only a brief survey without proofs of some results on the existence of a dense set of pairs $(x, x^*) \in X \times X^*$ each having a single weak orbit tending to infinity under every operator of a specific sequence of operators in $B(X)$. In this paper we are going to give an appropriate generalization for n −tuple weak orbits only of the results in [5] that do not involve any requirements upon specific subsets of the spectra of the operators.

2. Preliminary Results

Lemma 2.1. ([2, Lemma V.37.15]) *Let* $\varepsilon > 0$ *and* $(a_n)_{n\geq 1}$ *be a sequence of positive numbers satisfying* $\sum_{n\geq 1} a_n < \varepsilon$. *Then there is a sequence of positive numbers* $(b_n)_{n\geq 1}$ *such that* $b_n \to \infty$ *as* $n \to \infty$ *and* $\sum_{n\geq 1} a_n b_n < \varepsilon$.

Theorem 2.2. ([2, Theorem V.39.5]) *Let X and Y be Banach spaces and $(T_n)_{n\geq 1}$ be a sequence of operators in $B(X,Y)$. Let $(a_n)_{n\geq 1}$ be sequence of positive numbers with $\sum_{n\geq 1} a_n^{1/2} < \infty$. Then there are $x \in X$ and $y^* \in Y^*$ such that*

$$|\langle T_n x, y^* \rangle| \geq a_n \|T_n\|, \text{ for all } n \geq 1.$$

Moreover, given balls $B \subset X$ and $B^* \subset Y^*$ of radii greater than $\sum_{n\geq 1} a_n^{1/2} < \infty$, then it is possible to find $x \in B$ and $y^* \in B^*$ with this property.

Corollary 2.3. ([2, Corollary V.39.6]) *Let X and Y be Banach spaces and $(T_n)_{n\geq 1}$ be a sequence of operators in $B(X,Y)$ satisfying $\sum_{n=1}^{\infty} \|T_n\|^{-1/2} < \infty$. Then there exist $x \in X$ and $y^* \in Y^*$ such that $|\langle T_n x, y^* \rangle| \to \infty$. Moreover, the set of such pairs (x, y^*) is dense in $X \times Y^*$.*

3. Main Results for N-Tuple Weak Orbits

Let $F = \{1,2,\ldots,N\}$ for some $N \in \mathbb{N}$, $N \geq 2$, or $F = \mathbb{N}$.

Theorem 3.1. *If X is a Banach space, $\{T_i : i \in F\} \subset B(X)$ and $\left\{(a_{i,j})_{j\geq 1} : i \in F\right\}$ is a family of sequences of positive numbers such that $\sum_{i\in F, j\geq 1} a_{i,j}^{1/2} < \infty$, then for every open ball $B \subset X$ and every open ball $B^* \subset X^*$ with radii strictly larger then $\sum_{i\in F, j\geq 1} a_{i,j}^{1/2}$ there are $x \in B$ and $x^* \in B^*$ such that*

$$\left|\langle T_i^k x, x^* \rangle\right| \geq a_{i,k} \left\|T_i^k\right\|, \text{ for all } i \in F \text{ and } k \in \mathbb{N}.$$

Proof. Let $B \subset X$ and $B^* \subset X^*$ be open balls, each with radius strictly larger then $\sum_{i\in F, j\geq 1} a_{i,j}^{1/2}$ and put $T_{i,k} := T_i^k$, $i \in F$, $k \in \mathbb{N}$. Let $f: F \times \mathbb{N} \to \mathbb{N}$ be the bijective mapping defined with:

$$f(i,j)=\begin{cases} i+N(j-1), & \text{if } F=\{1,2,\ldots,N\},\\ \dfrac{(i+j-2)(i+j-1)}{2}, & \text{if } F=\mathbb{N},\end{cases}$$

and $g:\mathbb{N}\to F\times\mathbb{N}$ denote its inverse. If $(\alpha_n)_{n\geq 1}$ is a sequence of positive numbers and $(S_n)_{n\geq 1}$ is a sequence of operators defined with

$$\alpha_n = a_{g(n)} \text{ and } S_n = T_{g(n)}, \text{ for all } n\in\mathbb{N},$$

then $\sum_{n\geq 1}\alpha_n^{1/2}=\sum_{i\in F,j\geq 1}a_{i,j}^{1/2}<\infty$ and hence (by Theorem 2.2, applied on $(\alpha_n)_{n\geq 1}$, $(S_n)_{n\geq 1}$ and $X=Y$) there are $x\in B$ and $x^*\in B^*$ such that

$$|\langle S_n x, x^*\rangle| \geq \alpha_n\|S_n\|, \text{ for all } n\geq 1.$$

Given $(i,k)\in F\times\mathbb{N}$ and $n=f(i,k)$, these inequalities, along with the definition of $(\alpha_n)_{n\geq 1}$ and $(S_n)_{n\geq 1}$, will give

$$\left|\langle T_i^k x, x^*\rangle\right| = \left|\langle T_{i,k}x,x^*\rangle\right| = \left|\langle T_{g(n)}x,x^*\rangle\right| \geq a_{g(n)}\left\|T_{g(n)}\right\| = a_{i,k}\left\|T_i^k\right\|,$$

which completes the proof. ■

Theorem 3.2. *If* X *is Banach space and* $\{T_i: i\in F\}\subset B(X)$ *is a family of operators such that* $\sum_{k=1}^{\infty}\left\|T_i^k\right\|^{-1/2}<\infty$, *for all* $i\in F$, *then there is a dense set* $D\subset X\times X^*$ *such that the weak orbit* $\mathrm{Orb}(T_i,x,x^*)$ *tends to infinity for every pair* $(x,x^*)\in D$ *and every* $i\in F$. *If, in addition,* $\{T_i: i\in F\}$ *is a family of mutually commuting operators such that the sequences* $(T_i^k - T_j^k)_{k\geq 1}$ *are norm bounded for all* $i,j\in F$, *then for every* $n\in F$, *every* $1<m\leq n$ *and every pair* $(x,x^*)\in D$, *the* m*-tuple weak orbit* $\mathrm{Orb}\left(\{T_{i_j}\}_{j=1}^m, x, x^*\right)$ *tends to infinity for all* $1\leq i_1< i_2<\cdots<i_m\leq n$.

Proof. To prove the first assertion let $\varepsilon>0$, $B\subset X$ and $B^*\subset X^*$ be open balls each with a radius ε. For $i\in F$, let $\varepsilon_i>0$ be such that

$$\varepsilon_i\left(\sum_{k=1}^{\infty}\frac{1}{\left\|T_i^k\right\|^{1/2}}\right)<\frac{\varepsilon}{2^{i+1}},$$

and, by Lemma 2.1, let $(b_{i,k})_{k\geq 1}$ be the sequence of positive numbers such that $b_{i,k} \to \infty$ as $k \to \infty$ and

$$\sum_{k=1}^{\infty} \frac{\varepsilon_i b_{i,k}}{\left\|T_i^k\right\|^{1/2}} < \frac{\varepsilon}{2^{i+1}}. \tag{3.1}$$

For $(i,k) \in F \times \mathbb{N}$, put $a_{i,k} = \varepsilon_i^2 b_{i,k}^2 \left\|T_i^k\right\|^{-1}$. Then by (3.1) we have

$$\sum_{i\in F, k\geq 1} a_{i,k}^{1/2} = \sum_{i\in F}\sum_{k=1}^{\infty} \frac{\varepsilon_i b_{i,k}}{\left\|T_i^k\right\|^{1/2}} < \sum_{i\in F} \frac{\varepsilon}{2^{i+1}} < \frac{\varepsilon}{2}.$$

Hence, by Theorem 3.1, there are $x \in B$ and $x^* \in B^*$ such that

$$\left|\langle T_i^k x, x^*\rangle\right| \geq a_{i,k}\left\|T_i^k\right\| = \varepsilon_i^2 b_{i,k}^2 \left\|T_i^k\right\|^{-1}\left\|T_i^k\right\| = \varepsilon_i^2 b_{i,k}^2, \text{ for all } i \in F \text{ and } k \geq 1.$$

Letting $k \to \infty$, we obtain

$$\lim_{k\to\infty}\left|\langle T_i^k x, x^*\rangle\right| = \infty, \text{ for all } i \in F. \tag{3.2}$$

If, in addition, $\{T_i : i \in F\}$ is a family of mutually commuting operators such that the sequence $(T_i^k - T_j^k)_{k\geq 1}$ is norm bounded for all $i, j \in F$, let $M_{i,j} > 0$ be such that $\left\|T_i^k - T_j^k\right\| \leq M_{i,j}$, for all $k \geq 0$, and let $(x, x^*) \in X \times X^*$ be a pair satisfying (3.2). We continue by induction.

Let $m = 2$ and $1 \leq i_1 < i_2 \leq n$. Then

$$\begin{aligned}
\left|\langle T_{i_1}^{k_1+k_2} x, x^*\rangle\right| &\leq \left|\langle T_{i_1}^{k_1+k_2} x - T_{i_1}^{k_1} T_{i_2}^{k_2} x, x^*\rangle\right| + \left|\langle T_{i_1}^{k_1} T_{i_2}^{k_2} x, x^*\rangle\right| \\
&= \left|\langle T_{i_1}^{k_1}\left(T_{i_1}^{k_2} - T_{i_2}^{k_2}\right) x, x^*\rangle\right| + \left|\langle T_{i_1}^{k_1} T_{i_2}^{k_2} x, x^*\rangle\right| \\
&\leq \left\|T_{i_1}^{k_1}\right\| \cdot \left\|T_{i_1}^{k_2} - T_{i_2}^{k_2}\right\| \cdot \|x\| \cdot \|x^*\| + \left|\langle T_{i_1}^{k_1} T_{i_2}^{k_2} x, x^*\rangle\right| \\
&\leq \left\|T_{i_1}\right\|^{k_1} \cdot M_{i_1,i_2} \cdot \|x\| \cdot \|x^*\| + \left|\langle T_{i_1}^{k_1} T_{i_2}^{k_2} x, x^*\rangle\right|.
\end{aligned}$$

Since $\left|\langle T_{i_1}^{n} x, x^*\rangle\right| \to \infty$ as $n \to \infty$ (hence $\left|\langle T_{i_1}^{k_1+k_2} x, x^*\rangle\right| \to \infty$ as $k_2 \to \infty$, for all $k_1 \geq 0$), the above inequalities imply that

$$\left|\langle T_{i_1}^{k_1} T_{i_2}^{k_2} x, x^*\rangle\right| \to \infty, \text{ as } k_2 \to \infty, \text{ for all } k_1 \geq 0.$$

Similarly,

$$\begin{aligned}\left|\langle T_{i_2}^{k_1+k_2}x,x^*\rangle\right| &\le \left|\langle T_{i_2}^{k_1+k_2}x - T_{i_1}^{k_1}T_{i_2}^{k_2}x,x^*\rangle\right| + \left|\langle T_{i_1}^{k_1}T_{i_2}^{k_2}x,x^*\rangle\right| \\ &= \left|\langle T_{i_2}^{k_2}\left(T_{i_2}^{k_1} - T_{i_1}^{k_1}\right)x,x^*\rangle\right| + \left|\langle T_{i_1}^{k_1}T_{i_2}^{k_2}x,x^*\rangle\right| \\ &\le \left\|T_{i_2}^{k_2}\right\|\cdot\left\|T_{i_2}^{k_1} - T_{i_1}^{k_1}\right\|\cdot\|x\|\cdot\|x^*\| + \left|\langle T_{i_1}^{k_1}T_{i_2}^{k_2}x,x^*\rangle\right| \\ &\le \left\|T_{i_2}\right\|^{k_2}\cdot M_{i_2,i_1}\cdot\|x\|\cdot\|x^*\| + \left|\langle T_{i_1}^{k_1}T_{i_2}^{k_2}x,x^*\rangle\right|,\end{aligned}$$

imply that

$$\left|\langle T_{i_1}^{k_1}T_{i_2}^{k_2}x,x^*\rangle\right| \to \infty, \text{ as } k_1 \to \infty, \text{ for all } k_2 \ge 0.$$

To complete the proof, it is enough to show that the claim is true for $m = n$, under an inductive assumption that the $(n-1)$ −tuple weak orbit $\mathrm{Orb}\left(\{T_{i_j}\}_{j=1}^{n-1},x,x^*\right)$ tends to infinity for all $1 \le i_1 < i_2 < \cdots < i_{n-1} \le n$. For fixed $i \in \{1,2,\dots,n\}$ and $k_1,\ k_2,\dots,\ k_n \ge 0$ and arbitrary $j \in \{1,2,\dots,n\} \setminus \{i\}$, we have

$$\begin{aligned}&\left|\langle T_1^{k_1}\dots T_{i-1}^{k_{i-1}}T_j^{k_i}T_{i+1}^{k_{i+1}}\dots T_n^{k_n}x,x^*\rangle\right| \\ &\le \left|\langle T_1^{k_1}\dots T_{i-1}^{k_{i-1}}T_j^{k_i}T_{i+1}^{k_{i+1}}\dots T_n^{k_n}x - T_1^{k_1}T_2^{k_2}\dots T_n^{k_n}x,x^*\rangle\right| \\ &+ \left|\langle T_1^{k_1}T_2^{k_2}\dots T_n^{k_n}x,x^*\rangle\right| \\ &= \left|\langle T_1^{k_1}\dots T_{i-1}^{k_{i-1}}T_{i+1}^{k_{i+1}}\dots T_n^{k_n}\left(T_j^{k_i} - T_i^{k_i}\right)x,x^*\rangle\right| + \left|\langle T_1^{k_1}T_2^{k_2}\dots T_n^{k_n}x,x^*\rangle\right| \\ &\le \left\|T_1^{k_1}\dots T_{i-1}^{k_{i-1}}T_{i+1}^{k_{i+1}}\dots T_n^{k_n}\right\|\cdot\left\|T_j^{k_i} - T_i^{k_i}\right\|\cdot\|x\|\cdot\|x^*\| \\ &+ \left|\langle T_1^{k_1}T_2^{k_2}\dots T_n^{k_n}x,x^*\rangle\right| \\ &\le \left(\prod_{\substack{p=1\\ p\ne i}}^{n}\left\|T_p\right\|^{k_p}\right)\cdot M_{j,i}\cdot\|x\|\cdot\|x^*\| + \left|\langle T_1^{k_1}T_2^{k_2}\dots T_n^{k_n}x,x^*\rangle\right|.\end{aligned}$$

Since $j \in \{1,2,\dots,n\} \setminus \{i\}$,

$$\langle T_1^{k_1}\dots T_{i-1}^{k_{i-1}}T_j^{k_i}T_{i+1}^{k_{i+1}}\dots T_n^{k_n}x,x^*\rangle \in Orb(\{T_1,\dots,T_{i-1},T_{i+1},\dots,T_n\},x,x^*),$$

and since, by the inductive assumption, this $(n-1)$ −tuple weak orbit tents to infinity, we have

$$\left|\langle T_1^{k_1}\dots T_{i-1}^{k_{i-1}}T_j^{k_i}T_{i+1}^{k_{i+1}}\dots T_n^{k_n}x,x^*\rangle\right| \to \infty, \text{ as } k_i \to \infty, \text{ for all } k_j \ge 0, j \ne i.$$

This, together with the above inequalities implies that

$$\left|\langle T_1^{k_1} T_2^{k_2} \ldots T_n^{k_n} x, x^* \rangle\right| \to \infty, \text{ as } k_i \to \infty, \text{ for all } k_j \geq 0, j \neq i.$$

which completes the proof. ■

Corollary 3.3. *If X is Banach space and $\{T_i : i \in F\} \subset B(X)$ is a family of operators such that $r(T_i) > 1$ for all $i \in F$, then there is a dense set $D \subset X \times X^*$ such that the weak orbit $\mathrm{Orb}(T_i, x, x^*)$ tends to infinity for every pair $(x, x^*) \in D$ and every $i \in F$. If, in addition, $\{T_i : i \in F\}$ is a family of mutually commuting operators such that the sequences $(T_i^k - T_j^k)_{k \geq 1}$ are norm bounded for all $i, j \in F$, then for every $n \in F$, every $1 < m \leq n$ and every pair $(x, x^*) \in D$, the m −tuple weak orbit $\mathrm{Orb}\left(\{T_{i_j}\}_{j=1}^m, x, x^*\right)$ tends to infinity for all $1 \leq i_1 < i_2 < \cdots < i_m \leq n$.*

Proof. By Theorem 3.2 it is enough to show that an operator $T \in B(X)$ with spectral radius $r(T) > 1$ satisfies: $\sum_{n=1}^{\infty} \|T^n\|^{-1/2} < \infty$. If $r(T) > 1$, then there is $\lambda \in \sigma(T)$ such that $|\lambda| > 1$. Clearly $|\lambda|^{1/2} > 1$ and hence the series $\sum_{n=1}^{\infty} |\lambda|^{-n/2}$ converges. On the other hand, by the Spectral Mapping Theorem, $\lambda^n \in \sigma(T^n)$ for all $n \in \mathbb{N}$. Hence $|\lambda|^n \leq r(T^n) \leq \|T^n\|$ and $\sum_{n=1}^{\infty} \|T^n\|^{-1/2} \leq \sum_{n=1}^{\infty} |\lambda|^{-n/2} < \infty$. ■

4. Few Remarks on N-Tuple Orbits Tending to Infinity

The inequalities of form

$$\left|\langle T_1^{k_1} \ldots T_n^{k_n} x, x^* \rangle\right| \leq \left\|T_1^{k_1} \ldots T_n^{k_n} x\right\| \cdot \|x^*\|, (x, x^*) \in X \times X^*, k_i \geq 0, 1 \leq i \leq n,$$

clearly imply that the n −tuple orbit $\mathrm{Orb}(\{T_i\}_{i=1}^n, x)$ tends to infinity whenever there is $x^* \in X^*$ such that the n −tuple weak orbit $\mathrm{Orb}(\{T_i\}_{i=1}^n, x, x^*)$ tends to infinity. Hence, in the light of the results from the previous section, we can state the following alternative results for n −tuple orbits tending to infinity to a part of the results in [7].

Theorem 4.1. *If X is Banach space and $\{T_i : i \in F\} \subset B(X)$ is a family of operators such that $\sum_{k=1}^{\infty} \left\|T_i^k\right\|^{-1/2} < \infty$, for all $i \in F$, then there is a dense set $D \subset X$ such that the orbit*

$\mathrm{Orb}(T_i, x)$ *tends to infinity for every* $x \in D$ *and every* $i \in F$. *If, in addition,* $\{T_i : i \in F\} \subset B(X)$ *is a family of mutually commuting operators such that the sequences* $(T_i^k - T_j^k)_{k\geq 1}$ *are norm bounded for all* $i, j \in F$, *then for every* $n \in F$, *every* $1 < m \leq n$ *and every* $x \in D$, *the* m *−tuple orbit* $\mathrm{Orb}(\{T_{i_j}\}_{j=1}^m, x)$ *tends to infinity for all* $1 \leq i_1 < i_2 < \cdots < i_m \leq n$.

Corollary 4.2. *If* X *is Banach space and* $\{T_i : i \in F\} \subset B(X)$ *is a family of operators such that* $r(T_i) > 1$ *for all* $i \in F$, *then there is a dense set* $D \subset X$ *such that the orbit* $\mathrm{Orb}(T_i, x)$ *tends to infinity for every* $x \in D$ *and every* $i \in F$. *If, in addition,* $\{T_i : i \in F\} \subset B(X)$ *is a family of mutually commuting operators such that the sequences* $(T_i^k - T_j^k)_{k\geq 1}$ *are norm bounded for all* $i, j \in F$, *then for every* $n \in F$, *every* $1 < m \leq n$ *and every* $x \in D$ *the* m *−tuple orbit* $\mathrm{Orb}(\{T_{i_j}\}_{j=1}^m, x)$ *tends to infinity for all* $1 \leq i_1 < i_2 < \cdots < i_m \leq n$.

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