For the Fourier transform of the convolution in $\mathcal{D}'$ and Z'

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Abstract. In this paper we give another proof of the known lemma considering the Fourier transform of the convolution of a distribution and a function. Also, we give its application in the mentioned spaces.

1. Introduction

L.Schwarts has considered the Fourier transform of distributions in S'. The space S' has the important property that the Fourier transform of distribution in S' is, also, distribution in S'.
It is possible to define the Fourier transform of distributions in $\mathcal{D}'$ by introducing the spaces Z and Z'.
We will give some background of the spaces Z and Z'.

Definition 1. With Z we denote the space of all entire functions $\psi(z)$ for which there exists constants $C_{m,p}$ and a , such that the following condition

$$|z|^p |D^m\psi(z)| \le C_{m,p} \exp(a|y|),$$

holds for all $z = x + iy \in \mathbb{C}$ and, where p, m are nonnegative integers.

The sequence (ψ_n) of functions of Z converges to the function ψ in the space Z if the following 3 conditions hold:

(i) The sequence (ψ_n) converges uniformly to the function ψ on every compact subset of the complex plane;

(ii) For every m>0, the sequence $(D^m\psi_n)$ converges uniformly to the function $D^m\psi$ on every compact subset of the complex plane;

(iii) There exists constants $C_{m,p}$ and a , independent of n, such that

$$|z|^p |D^m\psi_n(z)| \le C_{m,p} \exp(a|y|),$$

for every n and $z = x + iy \in \mathbb{C}$.

The condition (i) says that the restriction of the function ψ of Z on the real line is an elements of the space S. In fact this set of functions is a proper subset of S and it is dense in S.

The space of all continuous linear functional on Z is denote by Z'.

It is easy to verify that if $\varphi \in \mathcal{D}$ then the Fourier transform is

$$F(\varphi, z) = \int_{\mathbb{R}} \varphi(t) e^{itz} dt$$

and it is an element of Z.

Also, for $\psi \in Z$, the inverse Fourier transform is

$$F^{-1}(\psi, z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \psi(t)e^{-itz}dt$$

and, it is an elements of the space $\mathcal{D}$.

Furthermore, if the sequence (φ_n) converges to the function φ in $\mathcal{D}$ then the sequence of the Fourier transforms $(F(\varphi_n))$ converges to the Fourier transform of the boundary function i.e. to $F(\varphi)$ in Z. Similarly, if the sequence of functions (ψ_n) converges to the function ψ in Z then the sequence of the inverse Fourier transforms $(F^{-1}(\psi_n))$ converges to the inverse Fourier transform of the boundary function i.e. to $F^{-1}(\psi)$ in $\mathcal{D}$.

Now, we give some background on the Fourier transforms of the distributions in the space $\mathcal{D}'$.

Definition 2. For $T \in \mathcal{D}'$, the Fourier and the inverse Fourier transform are defined by the formulas

$$\langle F(T), \psi\rangle = \langle T, F(\psi, t)\rangle \text{ and } \langle F^{-1}(T), \psi\rangle = \langle T, F^{-1}(\psi, t)\rangle \text{ for } \psi \in Z.$$

The Fourier transform of a distribution in $\mathcal{D}'$ is a distribution in Z'.

Definition 3. If $S \in Z'$ then the Fourier and the inverse Fourier transform are defined as follow

$$\langle F(S), \varphi\rangle = \langle S, F(\varphi, z)\rangle \text{ and } \langle F^{-1}(S), \varphi\rangle = \langle S, F^{-1}(\varphi, z)\rangle \text{ for } \varphi \in \mathcal{D}.$$

The Fourier transform of a distribution in Z' is a distribution in $\mathcal{D}'$.

1. If (f_n) is a sequence in S such that $f_n \to f$ in S then that sequence of Fourier transforms $(F(f_n))$ converges uniformly to $F(f)$ on R.

Note that $F(f) \in S$ if $f \in S$.

2. $D \subset S$ is dense and Z is dense in S.

Theorem If $T \in D'$ is arbitrary distribution then $T = \sum_{j=1}^{\infty} T_j$ where each T_j has compact support and hold the following two conditions:

a) Any compact subset of the real line intersect with supports of only finitely many supports of T_j.

b) $\lim_{N\to\infty}\sum_{j=1}^{N}\langle T_j, \phi\rangle = \langle T, \phi\rangle$ for all $\phi \in D$.

Theorem The Fourier transform is a continuous linear mapping of D' and Z'. Hence if the series of distributions $\sum T_k$ where $T_k \in D'$ for $k = 1, 2, 3, ...$ converges to the distribution T in D' then the series of its Fourier transforms converges in Z

Example. Let $T \in \mathcal{D}'$ has compact support, then its Fourier transform is

$$\left\langle F(T), \varphi \right\rangle = \left\langle T_t, \int_{\mathbb{R}} \varphi(x) e^{itx} dx \right\rangle = \left\langle T_t, \hat{\varphi} \right\rangle = \left\langle \left\langle T_t, e^{i\omega t} \right\rangle, \varphi \right\rangle.$$

Hence $F(T) = \left\langle T_t, e^{i\omega t} \right\rangle$. In particular if $T = \delta$ then $F(T) = \left\langle \delta_t, e^{i\omega t} \right\rangle = 1$.

2. Main results

2.1. We give another interesting proof of the following lemma:

Lemma 1. Let $T \in \mathcal{D}'$ be with compact support and let $\varphi \in S$. Then the Fourier transform of the convolution of distribution T and the function φ is equal to the product of their Fourier transforms i.e.

$$F(T * \varphi, \omega) = F(T, \omega) \cdot F(\varphi, \omega)$$

Proof. Since T has compact support , T is tempered distribution and the convolution $T * \varphi$ is a function of the space S. Also we know that the function of S has Fourier transform i.e.

$$F(T * \varphi, \omega) = \int_{\mathbb{R}} (T * \varphi)(x) e^{i\omega x} dx \qquad (1)$$

Since the integral of the right side of (1) is a Riemann integral, we may write it in the following form

$$\int_{\mathbb{R}} \left\langle T_t, \varphi(x-t) \right\rangle e^{i\omega x} dx = \lim_{N \to \infty} \int_{-N}^{N} \left\langle T_t, \varphi(x-t) \right\rangle e^{i\omega x} dx$$

for $N = 1, 2, 3, \ldots$

The function $f(x) = \left\langle T_t, \varphi(x-t) \right\rangle e^{i\omega x}$ is continuous and by the first integral mean value theorem, it follows that that there exists a point $x_N \in [-N, N]$ such that

$$\int_{-N}^{N} \left\langle T_t, \varphi(x-t) \right\rangle e^{i\omega x} dx = 2N \left\langle T_t, \varphi(x_N - t) e^{i\omega x_N} \right\rangle.$$

Now, we consider the sequences of functions $\left(f_N(t) \right)$, where

$$f_N(t) = 2N \varphi(x_N - t) e^{i\omega x_N} = \int_{-N}^{N} \varphi(x-t) e^{i\omega x} dx.$$

We will show that the sequence $\left(f_N(t) \right)$ is uniformly bounded and equicontinuous. Since

$$|f_N(t)| = \left|\int_{-N}^{N} \varphi(x-t)e^{i\omega x}dx\right| \le \int_{-N}^{N} |\varphi(x-t)|dx \le \|\varphi\|_1$$

we have that $(f_N(t))$ is uniformly bounded sequence.

Now, let $\varepsilon > 0$ be a given number and $t', t'' \in [-N, N]$ be points such that $|t' - t''| < \delta$ for some $\delta > 0.$ Then

$$|f_N(t'') - f_N(t')| = \left|\int_{-N}^{N} [\varphi(x-t'') - \varphi(x-t')]e^{i\omega x}dx\right| \le \int_{-N}^{N} |\varphi(x-t'') - \varphi(x-t')|\,dx.$$

By Theorem 9.5 ([6] pg.182), for a given $\varepsilon > 0$ there exists $\delta > 0$ such that for all $t', t'' \in [-N, N]$ for which $|t'' - t'| < \delta$, it holds

$$\int_{-N}^{N} |\varphi(x-t'') - \varphi(x-t')|\,dx < \varepsilon$$

Thus the sequence $(f_N(t))$ is eqiucontinuous.

Since

$$\lim_{N\to\infty} f_N(t) = \int_{-\infty}^{\infty} \varphi(x-t)e^{i\omega x}dx,$$

the Arzela Ascoli theorem asserts that the sequence $(f_N(t))$ converges uniformly on every compact subset of $\mathbb{R}$ to the function

$$\int_{\mathbb{R}} \varphi(x-t)e^{i\omega x}dx.$$

The same is true for every sequence $(f_N^{(k)}(t))$. Thus, we have shown that the sequence $(f_N(t))$ converges to the function

$$\int_{-\infty}^{\infty} \varphi(x-t)e^{i\omega x}dx \text{ in E.}$$

Since T is continuous linear functional in the space E, it implies that the sequences $\langle T_t, f_N(t)\rangle$ converges to the function $\left\langle T_t, \int_{-\infty}^{\infty} \varphi(x-t)e^{i\omega x}dx \right\rangle$

If we set
$u = x - t$, then

$$\left\langle T_t, \int_{-\infty}^{\infty} \varphi(x-t)e^{i\omega x}dx \right\rangle = \lim_{N\to\infty} \langle T_t, f_N(t)\rangle = \left\langle T_t, e^{i\omega t}\int_{\mathbb{R}} \varphi(u)e^{i\omega u}du \right\rangle =$$

$$= \langle T_t, e^{i\omega t}\rangle \cdot \int_{\mathbb{R}} \varphi(u)e^{i\omega u}du = F(T,\omega)\cdot F(\varphi,\omega).$$

Thus the proof is complete.

Now we give two corrolaries of the above lemma.

Theorem 1. Let $T \in D'$ have compact support and let (φ_k) be a sequence in S such that $\varphi_k \to \varphi$ in S.

Then it holds

$$\lim_{k\to\infty} F\left(T*\varphi_k,\omega\right) = \lim_{k\to\infty} F\left(T,\omega\right)\cdot F\left(\varphi_k,\omega\right) = F\left(T,\omega\right)\cdot F\left(\varphi,\omega\right).$$

Proof. The proof is similar to the proof of the Lemma above. Since $T*\varphi_k$ belongs to the space S, for every $k=1,2,3,...$, it has Fourier transform.

Thus

$$F\left(T*\varphi_k;\omega\right) = \int_{\mathbb{R}} \left(T*\varphi_k\right)(x)e^{i\omega x}dx \quad \text{for } k=1,2,3,...$$

We will show that

$$\lim_{k\to\infty} F\left(T*\varphi_k;\omega\right) = F\left(T;\omega\right)\cdot\lim_{k\to\infty} F\left(\varphi_k;\omega\right) =$$

$$= F\left(T;\omega\right)\cdot F\left(\varphi;\omega\right).$$

So, we have

$$\lim_{k\to\infty} F\left(T*\varphi_k;\omega\right) = \lim_{k\to\infty}\int_{\mathbb{R}} \left\langle T_t,\varphi_k(x-t)\right\rangle e^{i\omega x}dx =$$

$$\lim_{k\to\infty}\lim_{N\to\infty}\int_{-N}^{N} \left\langle T_t,\varphi_k(x-t)\right\rangle e^{i\omega x}dx.$$

Now we consider the sequence $\left(f_{N,k}(t)\right)$, where

$$f_{N,k}(t) = 2N\varphi_k(x_N - t)e^{i\omega x_N}$$

$$= \int_{-N}^{N} \varphi_k(x-t)e^{i\omega x}dx.$$

The sequence $\left(f_{N,k}\right)$ is uniformly bounded and equicontinuous.

Similarly, we can prove that the above holds for all derivatives $\left(f_{N,k}{}^{(p)}\right)$. Thus the sequence $\left(f_{N,k}\right)$ converges to the function $\int_{-\infty}^{\infty}\varphi_k(x-t)e^{i\omega x}dx$ in E.

Finally, if we take limit as $k\to\infty$, we get

$$\begin{aligned}&\lim_{k\to\infty}F\left(T*\varphi_k;\omega\right)=\\&\lim_{k\to\infty}\left(F\left(T;\omega\right)\cdot F(\varphi_k,\omega)\right)=\\&=F(T,\omega)\cdot\lim_{k\to\infty}F(\varphi_k,\omega)=\\&F(T,\omega)\cdot F(\varphi,\omega).\end{aligned}$$

So, the proof is complete.

Theorem 2. Let $T_k\in D'$ be distributions with compact support, let $\varphi\in S$ and suppose that $\lim_{k\to\infty}F\left(T_k*\varphi;\omega\right)$ and $\lim_{k\to\infty}F(T_k,\omega)\cdot F(\varphi,\omega)$ exist. Then

$$\lim_{k\to\infty}F\left(T_k*\varphi;\omega\right)=\lim_{k\to\infty}F(T_k,\omega)\cdot F(\varphi,\omega).$$

Proof. Since every T_k has compact support, for every $\varphi\in S$ the convolution $T_k*\varphi$ belongs to S and hence it has the Fourier transform $F\left(T_k*\varphi;\omega\right)$, which also belongs to the space S. Thus from the above lemma, we have that

$$F\left(T_k*\varphi;\omega\right)=F(T_k,\omega)\cdot F(\varphi,\omega).$$

Since the sequence of Fourier transforms of ?S converges uniformly on $\mathbb{R}$? to Fourier transform of S?, by taking limit of the both sides we, get

$$\lim_{k\to\infty}F\left(T_k*\varphi;\omega\right)=\lim_{k\to\infty}F(T_k,\omega)\cdot F(\varphi,\omega).$$

2.2. Application of the given lemma

Now we give the definition of the convolution of distributions and apply it the given lemma.

Definition 4. Let S and T be two distributions of $\mathcal{D}'$ and let T has compact support. Then the convolution is given by the formula

$$T*S=F^{-1}\left(F(S)\cdot F(T)\right)=2\pi F\left(F^{-1}(S)\cdot F^{-1}(T)\right). \tag{2}$$

Since $F(S)$ is in Z' and $F(T)$ is a multiplier in Z we have that the convolution $T*S$ belongs to the space $\mathcal{D}'$.

Note that in some books as in [2] or in [8] the convolution is defined in a different way, but the definitions are equivalent with (2).

In [4] is defined convolution for a larger class of distributions.

Now let $T \in \mathcal{D}'$ has compact support and let $\varphi \in S$. Since the function φ defines a tempered distribution $[\varphi]$, then the convolution of T and $[\varphi]$ is

$$T * [\varphi] = T * \varphi = F^{-1}(F(T) \cdot F([\varphi]))$$

and, because of the lemma, for the Fourier transform we have

$$F\left(T * [\varphi]\right) = F(T) \cdot F([\varphi]).$$

Also

$$F^{-1}(T * [\varphi]) = 2\pi F^{-1}(T) F^{-1}([\varphi]).$$

Now we solve the problem given in [[1],pg.160].

Problem. Let $Q(\omega) = c(\omega - \omega_1)^{\mu_1} \cdots (\omega - \omega_g)^{\mu_g}$. Find the solution of the equation

$$Q(\omega) H = 0$$

in the space Z'.

Solution. Let $\psi \in Z$, then $Q(\omega)\psi(\omega)$ also belongs to Z, since $Q(\omega)$ is a multiplier in Z.

Thus

$$\left\langle Q(\omega) H, \psi(\omega) \right\rangle = \left\langle H(\omega), Q(\omega)\psi(\omega) \right\rangle.$$

Now,

$$\left\langle \delta(\omega - \omega_1), c(\omega - \omega_1)^{\mu_1} \cdots (\omega - \omega_g)^{\mu_g} \psi(\omega) \right\rangle =$$
$$c(\omega_1 - \omega_1)^{\mu_1} (\omega_1 - \omega_2)^{\mu_2} \cdots (\omega_1 - \omega_g)^{\mu_g} \psi(\omega_1) = 0.$$

The same is true for

$$\delta^{(1)}(\omega - \omega_1), \delta^{(2)}(\omega - \omega_1), \ldots, \delta^{(\mu_1 - 1)}(\omega - \omega_1)$$

or $\delta^{(j)}(\omega - \omega_2)$ for $j = 0, 1, 2, \ldots, \mu_2 - 1$ and so on. Thus the solution has the form

$$H = \alpha_{11}(\omega - \omega_1) + \cdots + \alpha_{1\mu_1} \delta^{(\mu_1 - 1)}(\omega - \omega_1) + \cdots + \alpha_{k\mu_1} \delta(\omega - \omega_k) + \cdots + \alpha_{k\mu_k} \delta^{(\mu_k - 1)}(\omega - \omega_k)$$

As an application of the above problem we consider the homogenous linear differential equation with constant coefficients.

Let

$$a_n v^{(n)} + a_{n-1} v^{(n-1)} + \cdots + a_0 v = 0, \qquad (3)$$

where $a_n \neq 0$ for $n \geq 1$ are constants and v is allowed to be any distribution of $\mathcal{D}'$.

The differential equation (3) can be written in the form

$$\left(a_n\delta^{(n)} + a_{n-1}\delta^{(n-1)} + \cdots + a_0\delta\right) * v = 0.$$

If we apply the inverse Fourier transform then we obtain the equation

$$Q(\omega)F^{-1}(v) = 0.$$

where $F^{-1}(v)$ is in Z' which is the solution of the homogenous equation.

Thus $F^{-1}(v) = H$, hence $v = F(H) \in \mathcal{D}'$.

By the properties of the Fourier transform we get

$$v(t) = \alpha_{11}e^{i\omega_1 t} + \cdots + \alpha_1\mu_1(it)^{\mu_1 - 1}e^{i\omega_1 t} + \cdots + \alpha_{k1}e^{i\omega_k t} + \cdots + \alpha_k\mu_k(it)^{\mu_k - 1}e^{i\omega_k t}$$

Finally we have shown that the distributional solution of homogeneous equation in Z' is not other then the classical solution.

References

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