Some results concerning the analytic representation of convolution

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Abstract. In this paper we will prove some results concerning the analytic representation of the convolution of some functions.

1. Introduction

We use the standard notation from the Schwartz distribution theory.

The boundary value representation has been studied for a long time and from different points of view.

One of the first result is that if $f \in L^1$, then the function

$$\hat{f}(z) = \frac{1}{2\pi i} < f(t), \frac{1}{t-z} > \text{, for } \operatorname{Im} z \neq 0$$

is the Cauchy representation of f i.e.

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$$\lim_{y\to 0^+} < \hat{f}(x+iy) - \hat{f}(x-iy), \varphi(x) > = < f, \varphi >$$, for every $\varphi \in D$.

If $f, g \in L^1$ then $\int_{\mathbb{R}} \left| f(x-y)g(y) \right| dy < \infty$ for almost all x.

For these x, $h(x) = \int_{\mathbb{R}} f(x-y)g(y)dy$ belongs to $L^1(\mathbb{R})$ and $\left\| h \right\|_1 \leq \left\| f \right\|_1 \left\| g \right\|_1$. h is called the convolution of f and g and we write $h = f * g$. It is proven that the function $F(x,y) = f(x-y)g(y)$ Is a Borel function on $\mathbb{R}^2$, and that

$$\int_{\mathbb{R}} dy \int_{\mathbb{R}} \left| F(x,y) \right| dx = \int_{\mathbb{R}} \left| g(y) \right| dy \int_{\mathbb{R}} \left| f(x-y) \right| dx = \left\| f \right\|_1 \left\| g \right\|_1,$$

since $\int_{\mathbb{R}} \left| f(x-y) \right| dx = \left\| f \right\|_1$ for every $y \in \mathbb{R}^1$ by the translation-invariant of the Lebesque measure. Thus $F \in L^1(\mathbb{R}^2)$ and Fubini's Theorem implies that the integral $h(x) = \int_{\mathbb{R}} f(y)g(x-y)dy$ exists for almost all $x \in \mathbb{R}^1$ and that $h \in L^1(\mathbb{R}^1)$. Finally

$$\left\| h \right\|_1 = \int_{\mathbb{R}} \left| h(x) \right| dx \leq \int_{\mathbb{R}} dx \int_{\mathbb{R}} \left| F(x,y) \right| dy = \int_{\mathbb{R}} dy \int_{\mathbb{R}} \left| F(x,y) \right| dx = \left\| f \right\|_1 \left\| g \right\|_1$$

If $f \in L^1(\mathbb{R})$ $g \in L^p(\mathbb{R})$ for $1 < p < \infty$ then for almost all $y \in \mathbb{R}^1$, the functions of y, $f(x-y)g(y)$ and $f(y)g(x-y)$ are in $L^1(\mathbb{R})$. For all such x, we have $f * g = g * f$ a.e., $f * g \in L^p$ and $\left\| f * g \right\|_p \leq \left\| f \right\|_1 \left\| g \right\|_p$, where $(f * g)(x) = \int_{\mathbb{R}} f(x-y)g(y)dy$ and $(g * f)(x) = \int_{\mathbb{R}} g(x-y)f(y)dy$.

As above, $f(x-y)$, $g(y)$ and $h(x)$ are Borel function in $\mathbb{R}^2$, and so are their product taken two at a time and the function $f(x-y)\ g(y)\, h(x)$.

The proof that $h = f * g$ belongs to L^p for $1 < p < \infty$ is given in $[1]$.

2. Main results

We will prove some results concerning the analytic representat5ion of the convolution $h = f * g$ for $f,\ g \in L^1$ and $f \in L^1,\ g \in L^p$.

Theorem 1. Let f and g be in L^1 and let $h = g * f = f * g$. Then h has Cauchy representation

$$\hat{h}(z) = \frac{1}{2\pi i}\int \frac{h(t)}{t-z}dt = \int_{\mathbb{R}} f(t)\hat{g}(z-t)dt = \int_{\mathbb{R}} g(t)\hat{f}(z-t)dt, z = x+iy,\ \operatorname{Im} z \neq 0.$$

Proof. We have to show that

$$\lim_{y\to 0^+}\int_{\mathbb{R}}[\hat{h}(x+iy) - \hat{h}(x-iy)]\varphi(x)dx = < h,\varphi > \text{ , for } \varphi \in D.$$

$f,\ g \in L^1$ implies that $\hat{f}$ and $\hat{g}$ exist and $h \in L^1$. Consequently $\hat{h}$ exist. Thus

$$\lim_{y\to 0^+}\int_{\mathbb{R}}[\hat{h}(x+iy) - \hat{h}(x-iy)]\varphi(x)dx =$$
$$\int_{\mathbb{R}}(\frac{1}{2\pi i}\int_{\mathbb{R}}\frac{h(t)}{t-z}dt - \frac{1}{2\pi i}\int_{\mathbb{R}}\frac{h(t)}{t-\overline{z}})\varphi(x)dx =$$
$$\int_{\mathbb{R}}\varphi(x)dx\int_{\mathbb{R}}[\frac{1}{2\pi i}\int_{\mathbb{R}}\frac{f(u)g(u-t)}{t-z}du - \frac{1}{2\pi i}\int_{\mathbb{R}}\frac{f(u)g(u-t)}{t-\overline{z}}du]dt.$$

Since the integrals exist, Fubini's Theorem implies that

$$\lim_{y\to 0^+}\int_{\mathbb{R}}[\hat{h}(x+iy) - \hat{h}(x-iy)]\varphi(x)dx =$$
$$\int_{\mathbb{R}}\varphi(x)dx\int_{\mathbb{R}}f(u)du\frac{y}{\pi}\int_{\mathbb{R}}\frac{g(u-t)}{\left|t-z\right|^2}dt = \frac{y}{\pi}\int_{\mathbb{R}}\frac{\varphi(x)dx}{\left|t-z\right|^2}\int_{\mathbb{R}}f(u)du\int_{\mathbb{R}}g(u-t)dt =$$
$$\int_{\mathbb{R}}f(u)du\int_{\mathbb{R}}g(u-t)\overset{\wedge}{\varphi}(t+iy)dt.$$

Since $\overset{\wedge}{\varphi}(t+iy) \to \varphi(t)$ as $y \to 0^+$ uniformly on compact subset in the sense of D, we get that

$$\lim_{y\to 0^+} \int_{\mathbb{R}} [\hat{h}(x+iy) - \hat{h}(x-iy)]\varphi(x)dx =$$
$$\int_{\mathbb{R}} f(u)du \int_{\mathbb{R}} g(u-t)\varphi(t)dt =$$
$$\int_{\mathbb{R}} f(u)g(u-t)du \int_{\mathbb{R}} \varphi(t)dt =$$
$$\int_{\mathbb{R}} (f*g)(t)\varphi(t)dt = < f*g, \varphi > = < h, \varphi > .$$

So, the proof is complete.

Theorem 2. Let $f \in L^1(\mathbb{R})$ and $g \in L^p(\mathbb{R})$ and let $h = f*g$. Then h has the Cauchy representation $\hat{h}(z) = \frac{1}{2\pi i}\int \frac{h(t)}{t-z}dt$, $z = x+iy$, $\operatorname{Im} z \neq 0$.

Proof. For $\varphi \in D$,

$$\lim_{y\to 0^+} \int_{\mathbb{R}} [\hat{h}(x+iy) - \hat{h}(x-iy)]\varphi(x)dx =$$
$$\int_{\mathbb{R}} \frac{1}{2\pi i}[\int_{\mathbb{R}} (\frac{h(t)}{t-z} - \frac{h(t)}{t-\overline{z}})dt]\varphi(x)dx =$$
$$\int_{\mathbb{R}} \frac{1}{2\pi i}(\int_{\mathbb{R}}\int_{\mathbb{R}} [\frac{f(u)g(u-t)du}{t-z} - \frac{f(u)g(u-t)du}{t-\overline{z}}]dt)\varphi(x)dx.$$

The above integrals exist by the Hölder inequality, hence applying Fubini's theorem, we may change the order of integration and get that

$$\lim_{y\to 0^+} \int_{\mathbb{R}} [\hat{h}(x+iy) - \hat{h}(x-iy)]\varphi(x)dx =$$
$$\frac{1}{2\pi i}\int_{\mathbb{R}} (\frac{\varphi(x)}{t-z} - \frac{\varphi(x)}{t-\overline{z}})dx \int_{\mathbb{R}} f(u)du \int_{\mathbb{R}} g(u-t)dt =$$
$$\frac{y}{\pi}\int_{\mathbb{R}} \frac{\varphi(x)}{\left|t-z\right|^2}dx \int_{\mathbb{R}} f(u)du \int_{\mathbb{R}} g(u-t)dt.$$

Now by the Lema 5.4 [1], we get that $\frac{y}{\pi}\int_{\mathbb{R}} \frac{\varphi(x)}{\left|t-z\right|^2}dx = \overset{\wedge}{\varphi}(t+iy)$ and that

$$\int_{\mathbb{R}} f(u)du\int_{\mathbb{R}} g(u-t)\overset{\wedge}{\varphi}(t+iy)dt \text{ converges to } \int_{\mathbb{R}} f(u)du\int_{\mathbb{R}} g(u-t)\varphi(t)dt .$$

Finally, with one more use of Fubini's theorem, we get

$$\lim_{y\to 0^+}\int_{\mathbb{R}} [\hat{h}(x+iy)-\hat{h}(x-iy)]\varphi(x)dx =$$
$$\int_{\mathbb{R}} f(u)g(u-t)du\int_{\mathbb{R}}\varphi(t)dt = \int_{\mathbb{R}} (f*g)(t)\varphi(t)dt =< f*g,\varphi >.$$

We denote by $L_Q^p = \left\{ g \middle| g \text{ is a measurable function on } \mathbb{R} \text{ and } \frac{g}{Q}\in L^p \right\}$, where Q is a function without real roots.

Theorem 3. Suppose that $f\in L^1(\mathbb{R})$, Q is a function without real roots and g is measurable function on $\mathbb{R}$ that belongs to the space L_Q^p. The convolution of the functions $f\in L^1$ and $\frac{g}{Q}\in L^p$, $h = f*\left(\frac{g}{Q}\right)$, $h\in L^p$, $\left\|h\right\|_p \le \left\|f\right\|_1 \left\|g(Q)\right\|_p$ and h has Cauchy representation $\hat{h}(z) = \frac{1}{2\pi i} < h, \frac{1}{t-z} >$.

Proof. The fact that $h\in L^p$, $\left\|h\right\|_p \le \left\|f\right\|_1 \left\|g(Q)\right\|_p$ can be easily proven as in the introduction part.

Let $\varphi\in D$ be arbitrary function. Then we have

$$\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}} [\hat{h}(x+i\varepsilon)-\hat{h}(x-i\varepsilon)]\varphi(x)dx =$$
$$\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}} [\frac{1}{2\pi i}\int_{\mathbb{R}} (\frac{h(t)}{t-z}-\frac{h(t)}{t-\overline{z}})dt]\varphi(x)dx =$$
$$\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}} [\frac{1}{2\pi i}\int_{\mathbb{R}} \frac{dt}{t-z}\int_{\mathbb{R}} f(u)\frac{g(u-t)}{Q(u-t)}du - \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{dt}{t-\overline{z}}\int_{\mathbb{R}} f(u)\frac{g(u-t)}{Q(u-t)}du]\varphi(x)dx.$$

Since the integrals exist, by Fubini's theorem, we may change the order of integration and get

$$\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}}[\hat{h}(x+i\varepsilon)-\hat{h}(x-i\varepsilon)]\varphi(x)dx =$$
$$\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}}\varphi(x)dx\,\frac{y}{\pi}\int_{\mathbb{R}}\frac{dt}{\left|t-z\right|^2}\int_{\mathbb{R}}f(u)\frac{g(u-t)}{Q(u-t)}du =$$
$$\lim_{\varepsilon\to 0^+}\frac{y}{\pi}\int_{\mathbb{R}}\frac{\varphi(x)dx}{\left|t-z\right|^2}\int_{\mathbb{R}}f(u)\int_{\mathbb{R}}\frac{g(u-t)}{Q(u-t)}dudt =$$
$$\lim_{\varepsilon\to 0^+}\frac{y}{\pi}\int_{\mathbb{R}}\frac{\varphi(x)dx}{\left|t-z\right|^2}\int_{\mathbb{R}}f(u)\frac{g(u-t)}{Q(u-t)}dudt.$$

By the Lebesque dominated convergence theorem and the Lema 5.4 in [1], we have that

$$\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}}[\hat{h}(x+i\varepsilon)-\hat{h}(x-i\varepsilon)]\varphi(x)dx = \int_{\mathbb{R}}f(u)du\int_{\mathbb{R}}\frac{g(u-t)}{Q(u-t)}\varphi(t)dt..$$

One more application of Fubini's theorem gives that

$$\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}}[\hat{h}(x+i\varepsilon)-\hat{h}(x-i\varepsilon)]\varphi(x)dx =$$
$$\int_{\mathbb{R}}f(u)\frac{g(u-t)}{Q(u-t)}du\int_{\mathbb{R}}\varphi(t)dt =$$
$$\int_{\mathbb{R}}(f*\frac{g}{Q})(t)\varphi(t)dt =< f*\frac{g}{Q},\varphi >=< h,\varphi > .$$

Note. In similar way, it can be proven another version of this theorem. Namely, if $g \in L^p$ and if f is measurable function such that $f / p \in L^1$ then the convolution $(f / p) * g \in L^p$ and also as before it is proved that has Cauchy representation.

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