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N-TUPLE ORBITS TENDING TO INFINITY

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Abstract. In this paper we prove some results on the existence of a dense set of vectors each having an n-tuple orbit tending to infinity for sequences of mutually commuting bounded linear operators acting on an infinite dimensional complex Banach space.

1. INTRODUCTION

Let X be a complex infinite-dimensional Banach space and $B(X)$ the algebra of all bounded linear operators acting on X. For an operator $T \in B(X)$, $\sigma(T)$, $\sigma_{\mathrm{p}}(T)$, $\sigma_{\mathrm{ap}}(T)$ and $r(T)$ will denote the spectrum, the point spectrum, the approximate point spectrum and the spectral radius of the operator T, respectively.

If $T_1, T_2, \ldots, T_n \in B(X)$ are mutually commuting operators, the *n-tuple orbit* (or the orbit under the n-tuple $\mathbf{T} = (T_1, T_2, \ldots, T_n)$) of the vector $x \in X$ is the set

$$\mathrm{Orb}(\{T_i\}_{i=1}^{n}, x) = \mathrm{Orb}(\mathbf{T}, x) = \left\{ T_1^{k_1} T_2^{k_2} \ldots T_n^{k_n} x : k_i \geq 0; 1 \leq i \leq n \right\}. \tag{1.1}$$

The n-tuple orbit *tends to infinity* if

$$\lim_{k_i \to \infty} \left\| T_1^{k_1} T_2^{k_2} \ldots T_n^{k_n} x \right\| = \infty, \text{ for all } k_j \geq 0,\ j \neq i,\ 1 \leq i, j \leq n .$$

For $n = 1$, the n-tuple orbit (1.1) reduces to a simple sequence of form

$$\mathrm{Orb}(T, x) = \left\{ T^n x : n = 0, 1, 2, \ldots \right\},$$

usually referred as *single orbit* (or simply *orbit*) of the vector $x \in X$ under the operator T. Regardless of the dimension of the space, single orbits tending to infinity may exist only when T is power unbounded operator, i.e. when $\sup_n \|T^n\| = \infty$. In this case, by the Banach-Steinhaus theorem, the space will contain a dense G_δ -set of vectors each having an unbounded orbit under T (i.e. orbit with $\sup_n \|T^n x\| = \infty$). But, unlike the case of an operator T acting on a finite-dimensional space where the only unbounded orbits for T are those tending to infinity and may exist if, and only if, $r(T) > 1$, in the case of an

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infinite-dimensional space, the structure of the set of all vectors with orbits tending to infinity can be quite different. Clearly, if $\sigma_{\mathrm{p}}(T)$ contains a point λ such that $|\lambda|>1$, this set will contain all the elements of $\mathrm{Ker}(T-\lambda I)\setminus\{0\}$. Furthermore, the set of all vectors with orbits tending to infinity can be dense in the whole space, even if the point spectrum of the operator is empty. The results obtained by B. Beauzamy for operators on infinite-dimensional Hilbert or reflexive Banach space X (cf. [1, Ch. III]) imply that for any operator $T\in B(X)$ for which $\sigma_{\mathrm{ap}}(T)\setminus\sigma_{\mathrm{p}}(T)$ contains a point λ with $|\lambda|>1$, the space will contain a dense set D such that $\|T^n x\|\to\infty$ as $n\to\infty$, for all $x\in D$. The results obtained by V. Müller ([7] and [8]) imply that such set exists for any operator T on arbitrary infinite-dimensional Banach space as long as $r(T)>1$. In general, this set is not a G_δ-set since the space may contain another dense G_δ-set of vectors with unbounded orbits: vectors for which $\mathrm{Orb}(T,x)$ itself is dense in the whole space (cf. [9, Theorem 1] or [1, III.0.C]).

Under the assumption that T_1 and T_2 are bounded linear operators on infinite-dimensional Hilbert or reflexive Banach space satisfying

$$(\sigma_{\mathrm{ap}}(T_1)\setminus\sigma_{\mathrm{p}}(T_1))\cap\{\lambda\in\mathbb{C}:|\lambda|>1\}\neq\varnothing,$$
$$(\sigma_{\mathrm{ap}}(T_2)\setminus\sigma_{\mathrm{p}}(T_2))\cap\{\lambda\in\mathbb{C}:|\lambda|>1\}\neq\varnothing,$$

in [2] and [3] the authors have shown that the space contains a dense set D such that

$$\|T_1^n x\|\to\infty \text{ and } \|T_2^n x\|\to\infty, \text{ for all } x\in D.$$

If, in addition, T_1 and T_2 are commuting operators, each bounded bellow, then for every $x\in D$ the corresponding 2-tuple orbit tends to infinity ([10, Theorem 1.4]):

$$\left\|T_1^{k_1}T_2^{k_2}x\right\|\to\infty \text{ as } k_1\to\infty, \text{ for every } k_2\geq 0,$$

and

$$\left\|T_1^{k_1}T_2^{k_2}x\right\|\to\infty \text{ as } k_2\to\infty, \text{ for every } k_1\geq 0.$$

Using the following three results, in the next section we are going to generalize this result for *n*-tuple orbits and sequences of mutually commuting operators each bounded bellow.

Theorem 1.1. ([8, Theorem V.37.14]) *Let X and Y be Banach spaces and let $(T_n)_{n\geq 1}$ be a sequence of operators in $B(X,Y)$. Then for every sequence of positive numbers $(a_n)_{n\geq 1}$ with $\sum_{n\geq 1}a_n<\infty$, in every open ball in X with* *radius strictly larger than* $\sum_{n\geq 1} a_n < \infty$, *there is a vector* $x \in X$ *satisfying* $\|T_n x\| \geq a_n \|T_n\|$, *for all* $n \geq 1$.

Lemma 1.2. ([8, Lemma V.37.15]) *Let* $\varepsilon > 0$ *and* $(a_n)_{n\geq 1}$ *be a sequence of positive numbers satisfying* $\sum_{n\geq 1} a_n < \varepsilon$. *Then there is a sequence of positive numbers* $(b_n)_{n\geq 1}$ *such that* $b_n \to \infty$ *as* $n \to \infty$ *and* $\sum_{n\geq 1} a_n b_n < \varepsilon$.

Corollary 1.3. ([8, Corollary V.37.16]) *If* $T \in B(X)$ *satisfies* $\sum_{n=1}^{\infty} \|T^n\|^{-1} < \infty$, *then there is a dense set* $D \subset X$ *such that* $\mathrm{Orb}(T, x)$ *tends to infinity for every* $x \in D$.

2. Main Results

Throughout the rest of this paper we assume that the spaces are complex and infinite-dimensional.

Lemma 2.1. *Let* X *be a Banach space and* $T_1, T_2, \ldots T_n \in B(X)$ *are mutually commuting operators with at least one of the following properties:*

(P.1) *the operator* T_i *is bounded bellow, for every* i;

(P.2) $(T_i^k - T_j^k)_{k\geq 0}$ *is a norm bounded sequence, for every* i *and* j.

If $x \in X$ *is such that* $\mathrm{Orb}(T_i, x)$ *tends to infinity for every* $i \in \{1, 2, \ldots, n\}$, *then for every* $1 \leq m \leq n$ *and every* $1 \leq i_1 < i_2 < \ldots < i_m \leq n$ *the m-tuple orbit* $\mathrm{Orb}(\{T_{i_j}\}_{j=1}^{m}, x)$ *tends to infinity.*

Proof. If the operators $T_1, T_2, \ldots T_n$ have the property (P.1), then there are positive numbers $C_1, C_2, \ldots C_n$ such that

$$\|T_i x\| \geq C_i \|x\|, \text{ for all } x \in X,\ 1 \leq i \leq n.$$

Hence, if $1 \leq m \leq n$, $1 \leq i_1 < i_2 < \ldots < i_m \leq n$ and $k_j \geq 0$, $j \in \{1, 2, \ldots, m\}$, then for every $s \in \{1, 2, \ldots, m\}$ and fixed values for k_j, $j \in \{1, 2, \ldots, m\} \setminus \{s\}$

$$\left\|T_{i_1}^{k_1} T_{i_2}^{k_2} \ldots T_{i_m}^{k_m} x\right\| \geq \left(\prod_{\substack{l=1 \\ l \neq s}}^{m} C_{i_l}^{k_l} \right) \cdot \left\|T_{i_s}^{k_s} x\right\| \to \infty, \text{ as } k_s \to \infty.$$

Now, assume that the operators $T_1, T_2, \ldots T_n$ have the property (P.2). For $i, j \in \{1, 2, \ldots, n\}$, let $M_{i,j} > 0$ is such that $\left\| T_i^k - T_j^k \right\| \le M_{i,j}$, for every $k \ge 0$.

We continue by induction. Let $m = 2$ and $1 \le i_1 < i_2 \le n$. Then

$$\begin{aligned}\left\| T_{i_1}^{k_1+k_2} x \right\| &\le \left\| T_{i_1}^{k_1+k_2} x - T_{i_1}^{k_1} T_{i_2}^{k_2} x \right\| + \left\| T_{i_1}^{k_1} T_{i_2}^{k_2} x \right\| \\ &= \left\| T_{i_1}^{k_1} (T_{i_1}^{k_2} - T_{i_2}^{k_2}) x \right\| + \left\| T_{i_1}^{k_1} T_{i_2}^{k_2} x \right\| \\ &\le \left\| T_{i_1}^{k_1} \right\| \cdot \left\| T_{i_1}^{k_2} - T_{i_2}^{k_2} \right\| \cdot \|x\| + \left\| T_{i_1}^{k_1} T_{i_2}^{k_2} x \right\| \\ &\le \left\| T_{i_1} \right\|^{k_1} \cdot M_{i_1,i_2} \cdot \|x\| + \left\| T_{i_1}^{k_1} T_{i_2}^{k_2} x \right\|.\end{aligned}$$

Since $\left\| T_{i_1}^n x \right\| \to \infty$ as $n \to \infty$ (and hence $\left\| T_{i_1}^{k_1+k_2} x \right\| \to \infty$ as $k_2 \to \infty$, for all $k_1 \ge 0$), the above inequalities imply that

$$\left\| T_{i_1}^{k_1} T_{i_2}^{k_2} x \right\| \to \infty, \text{ as } k_2 \to \infty, \text{ for all } k_1 \ge 0.$$

Similarly, the following inequalities

$$\begin{aligned}\left\| T_{i_2}^{k_1+k_2} x \right\| &\le \left\| T_{i_2}^{k_1+k_2} x - T_{i_1}^{k_1} T_{i_2}^{k_2} x \right\| + \left\| T_{i_1}^{k_1} T_{i_2}^{k_2} x \right\| \\ &= \left\| T_{i_2}^{k_2} (T_{i_2}^{k_1} - T_{i_1}^{k_1}) x \right\| + \left\| T_{i_1}^{k_1} T_{i_2}^{k_2} x \right\| \\ &\le \left\| T_{i_2}^{k_2} \right\| \cdot \left\| T_{i_2}^{k_1} - T_{i_1}^{k_1} \right\| \cdot \|x\| + \left\| T_{i_1}^{k_1} T_{i_2}^{k_2} x \right\| \\ &\le \left\| T_{i_2} \right\|^{k_2} \cdot M_{i_2,i_1} \cdot \|x\| + \left\| T_{i_1}^{k_1} T_{i_2}^{k_2} x \right\|.\end{aligned}$$

imply that

$$\left\| T_{i_1}^{k_1} T_{i_2}^{k_2} x \right\| \to \infty, \text{ as } k_1 \to \infty, \text{ for all } k_2 \ge 0.$$

To complete the proof, it is enough to show the claim for $m = n$, under the assumption that $\mathrm{Orb}(\{T_{i_j}\}_{j=1}^{n-1}, x)$ tends to infinity for all $1 \le i_1 < \ldots < i_{n-1} \le n$.

For a fixed $i \in \{1, \ldots, n\}$, arbitrary $j \in \{1, \ldots, n\} \setminus \{i\}$ and $k_1, \ldots, k_n \ge 0$ we have

$$
\begin{aligned}
&\left\|T_1^{k_1}\dots T_{i-1}^{k_{i-1}}T_j^{k_i}T_{i+1}^{k_{i+1}}\dots T_n^{k_n}x\right\| \\
&\quad\le \left\|T_1^{k_1}\dots T_{i-1}^{k_{i-1}}T_j^{k_i}T_{i+1}^{k_{i+1}}\dots T_n^{k_n}x - T_1^{k_1}T_2^{k_2}\dots T_n^{k_n}x\right\| + \left\|T_1^{k_1}T_2^{k_2}\dots T_n^{k_n}x\right\| \\
&\quad= \left\|T_1^{k_1}\dots T_{i-1}^{k_{i-1}}T_{i+1}^{k_{i+1}}\dots T_n^{k_n}(T_j^{k_i}-T_i^{k_i})x\right\| + \left\|T_1^{k_1}T_2^{k_2}\dots T_n^{k_n}x\right\| \\
&\quad\le \left\|T_1^{k_1}\dots T_{i-1}^{k_{i-1}}T_{i+1}^{k_{i+1}}\dots T_n^{k_n}\right\|\cdot\left\|T_j^{k_i}-T_i^{k_i}\right\|\cdot\|x\| + \left\|T_1^{k_1}T_2^{k_2}\dots T_n^{k_n}x\right\| \\
&\quad\le \left(\prod_{\substack{l=1\\ l\ne i}}^{n}\|T_l\|^{k_l}\right)\cdot M_{i,j}\cdot\|x\| + \left\|T_1^{k_1}T_2^{k_2}\dots T_n^{k_n}x\right\|.
\end{aligned}
$$

Since $j\in\{1,2,\dots,n\}\setminus\{i\}$,

$$T_1^{k_1}\dots T_{i-1}^{k_{i-1}}T_j^{k_i}T_{i+1}^{k_{i+1}}\dots T_n^{k_n}x\in \mathrm{Orb}(\{T_1\dots T_{i-1}T_{i+1}\dots T_n\},x),$$

and, by assumption, this (n-1)-tuple orbit tents to infinity,

$$\left\|T_1^{k_1}\dots T_{i-1}^{k_{i-1}}T_j^{k_i}T_{i+1}^{k_{i+1}}\dots T_n^{k_n}x\right\|\to\infty \text{ as } k_i\to\infty, \text{ for all } k_j\ge 0,\ j\ne i.$$

This, together with the above inequalities implies that

$$\left\|T_1^{k_1}T_2^{k_2}\dots T_n^{k_n}x\right\|\to\infty \text{ as } k_i\to\infty, \text{ for all } k_j\ge 0,\ j\ne i,$$

which completes the proof. □

Theorem 2.2. *If X is a Banach space and $T_1,T_2,\dots T_n\in B(X)$ are operators with $r(T_i)>1$, $1\le i\le n$, then there is a dense set $D\subset X$ such that $\mathrm{Orb}(T_i,x)$ tends to infinity for every $x\in D$ and every $1\le i\le n$. If, in addition, the operators are mutually commuting and have at least one of the properties (P.1) and (P.2) in Lemma 2.1, then the m-tuple orbit $\mathrm{Orb}(\{T_{i_j}\}_{j=1}^{m},x)$ tends to infinity for every $x\in D$, $1\le m\le n$ and $1\le i_1<i_2<\dots<i_m\le n$.*

Proof. By Lemma 2.1, it is sufficient to prove the first assertion in the theorem.

Let $z\in X$ and $\varepsilon>0$. Since $r(T_i)>1$ there is $\lambda_i\in\sigma(T_i)$ such that $|\lambda_i|>1$, $1\le i\le n$. If $q,C\in\mathbb{R}$ are chosen such that

$$1<q<\min\{|\lambda_1|,|\lambda_2|,\dots,|\lambda_n|\},$$

and

$$0<C<\frac{\varepsilon(q-1)^2}{2(n+1)},$$

then the sequences of positive numbers $\{(a_{i,k})_{k\ge1}:1\le i\le n\}$ defined with

$$a_{i,k} = Cq^{-(i+k)},\ 1 \le i \le n,\ k \ge 1,$$

will satisfy

$$\sum_{1\le i\le n} \sum_{k\ge 1} a_{i,k} < \frac{\varepsilon}{2}. \tag{2.1}$$

If the sequence of operators $(S_j)_{j\ge 1}$ and the sequence of positive numbers $(a_j)_{j\ge 1}$ are defined with

$$S_{(k-1)n+i} = T_i^k \text{ and } a_{(k-1)n+i} = a_{i,k}, \text{ for } 1 \le i \le n,\ k \ge 1, \tag{2.2}$$

then

$$\sum_{j\ge 1} a_j = \sum_{1\le i\le n} \sum_{k\ge 1} a_{i,k},$$

and hence, by Theorem 1.1 (applied on $(S_j)_{j\ge 1}$ and $(a_j)_{j\ge 1}$), the Spectral Mapping Theorem and (2.1), the open ball with center z and radius ε will contain a vector $x \in X$ such that for every $1 \le i \le n$ and $k \ge 1$,

$$\begin{aligned} \left\|T_i^k x\right\| &= \left\|S_{(k-1)n+i} x\right\| \ge a_{(k-1)n+i} \left\|S_{(k-1)n+i}\right\| = a_{i,k} \left\|T_i^k\right\| \\ &\ge Cq^{-(i+k)} \left|\lambda_i\right|^k = Cq^{-i} \left|q^{-1}\lambda_i\right|^k. \end{aligned}$$

Since, by the choice of q, $|q^{-1}\lambda_i|^k \to \infty$ as $k \to \infty$, for every $1 \le i \le n$, the above inequalities imply that

$$\left\|T_i^k x\right\| \to \infty \text{ as } k \to \infty, \text{ for all } 1 \le i \le n,$$

which completes the proof. □

By Theorem 1.1 and Lemma 2.1 alone we can obtain similar result for sequence of operators $(T_i)_{i\ge 1}$ in $B(X)$.

Theorem 2.3. *If X is a Banach space and $(T_i)_{i\ge 1}$ is a sequence of operators in $B(X)$ for which there is $\beta > 0$ such that $r(T_i) > 1 + \beta$, for all $i \ge 1$, then there is a dense set $D \subset X$ such that* $\operatorname{Orb}(T_i, x)$ *tends to infinity for every $x \in D$ and $i \ge 1$. If, in addition, the operators are mutually commuting and have at least one of the properties (P.1) and (P.2) in Lemma 2.1, then for every $n \ge 1$ and every positive integers $i_1 < i_2 < \ldots < i_n$ the n-tuple orbit* $\operatorname{Orb}(\{T_{i_j}\}_{j=1}^n, x)$ *tends to infinity for every $x \in D$.*

The proof of the first assertion in Theorem 2.3 is given in [6].

The requirement "*there is* $\beta > 0$ *such that* $r(T_i) > 1+\beta$, *for all* $i \geq 1$" in Theorem 2.3 can be replaced with the following one: "$r(T_i) > 1$, *for all* $i \geq 1$". In order to show this, first we are going to give an appropriate generalization of Corollary 1.3.

Theorem 2.4. *If* X *is a Banach space and* $T_1, T_2, \ldots T_n \in B(X)$ *are operators satisfying* $\sum_{n=1}^{\infty} \left\| T_i^n \right\|^{-1} < \infty$, *for all* $1 \leq i \leq n$, *then there is a dense set* $D \subset X$ *such that* $\mathrm{Orb}(T_i, x)$ *tends to infinity for every* $x \in D$ *and every* $1 \leq i \leq n$. *If, in addition, the operators are mutually commuting and have at least one of the properties (P.1) and (P.2) in Lemma 2.1, then the m-tuple orbit* $\mathrm{Orb}(\{T_{i_j}\}_{j=1}^{m}, x)$ *tends to infinity for every* $x \in D$, $1 \leq m \leq n$ *and* $1 \leq i_1 < i_2 < \ldots < i_m \leq n$.

Proof. Once again, by Lemma 2.1, it is sufficient to prove the first assertion in the theorem. Let $z \in X$ and $\varepsilon > 0$. For $1 \leq i \leq n$, let $\varepsilon_i > 0$ be such that

$$\varepsilon_i \left(\sum_{k=1}^{\infty} \left\| T_i^k \right\|^{-1} \right) < \frac{\varepsilon}{2(n+1)}.$$

By Lemma 1.2 there are sequences of positive numbers $(b_{i,k})_{k \geq 1}$ so that $b_{i,k} \to \infty$ as $k \to \infty$ and

$$\sum_{k=1}^{\infty} \varepsilon_i b_{i,k} \left\| T_i^k \right\|^{-1} < \frac{\varepsilon}{2(n+1)}.$$

For $1 \leq i \leq n$ and $k \in \mathbb{N}$, let $a_{i,k} = \varepsilon_i b_{i,k} \left\| T_i^k \right\|^{-1}$. If the sequence of operators $(S_j)_{j \geq 1}$ and the sequence of positive numbers $(a_j)_{j \geq 1}$ are defined with (2.2), then $\sum_{j \geq 1} a_j < \varepsilon / 2$. Hence, by Theorem 1.1, there is a vector $x \in X$ satisfying $\|x - z\| < \varepsilon$ and for every $1 \leq i \leq n$ and $k \geq 1$,

$$\begin{aligned} \left\| T_i^k x \right\| &= \left\| S_{(k-1)n+i} x \right\| \\ &\geq a_{(k-1)n+i} \left\| S_{(k-1)n+i} \right\| = a_{i,k} \left\| T_i^k \right\| = \varepsilon_i b_{i,k} \left\| T_i^k \right\|^{-1} \left\| T_i^k \right\| = \varepsilon_i b_{i,k}. \end{aligned}$$

This implies that

$$\left\| T_i^k x \right\| \to \infty \text{ as } k \to \infty, \text{ for all } 1 \leq i \leq n,$$

which completes the proof. □

Theorem 2.5. *If* X *is a Banach space and* $(T_i)_{i\geq 1}$ *is a sequence of operators in* $B(X)$ *such that* $\sum_{k=1}^{\infty}\left\|T_i^k\right\|^{-1} < \infty$, *for all* $i \geq 1$, *then there is a dense set* $D \subset X$ *so that* $\mathrm{Orb}(T_i, x)$ *tends to infinity for every* $x \in D$ *and* $i \geq 1$. *If, in addition, the operators are mutually commuting and have at least one of the properties (P.1) and (P.2) in Lemma 2.1, then for every* $n \geq 1$ *and every positive integers* $i_1 < i_2 < \ldots < i_n$ *the n-tuple orbit* $\mathrm{Orb}(\{T_{i_j}\}_{j=1}^{n}, x)$ *tends to infinity for every* $x \in D$.

The proof of the first assertion in Theorem 2.5 the is given in [6].

Corollary 2.6. *If* $(T_i)_{i\geq 1}$ *is a sequence in* $B(X)$ *such that* $r(T_i) > 1$ *for all* $i \geq 1$, *then there is a dense set* $D \subset X$ *such that* $\mathrm{Orb}(T_i, x)$ *tends to infinity for every* $x \in D$ *and* $i \geq 1$. *If, in addition, the operators are mutually commuting and have at least one of the properties (P.1) and (P.2) in Lemma 2.1, then for every* $n \geq 1$ *and every positive integers* $i_1 < i_2 < \ldots < i_n$ *the n-tuple orbit* $\mathrm{Orb}(\{T_{i_j}\}_{j=1}^{n}, x)$ *tends to infinity for every* $x \in D$.

Proof. Let $i \in \mathbb{N}$. Since $r(T_i) > 1$ there is $\lambda_i \in \sigma(T_i)$ so that $|\lambda_i| > 1$. By the Spectral Mapping Theorem, for every $n \in \mathbb{N}$, $\lambda_i^n \in \sigma(T_i^n)$ and hence,

$$|\lambda_i|^n \leq r(T_i^n) \leq \left\|T_i^n\right\|.$$

This would imply that

$$\sum_{n=1}^{\infty}\left\|T_i^n\right\|^{-1} \leq \sum_{n=1}^{\infty}|\lambda_i|^{-n} < \infty.$$

Now the conclusion follows from Theorem 2.5. □

Having in mind that every invertible operator is bounded bellow, we have the following corollary.

Corollary 2.7. *If* $(T_i)_{i\geq 1}$ *is a sequence of invertible, mutually commuting operators in* $B(X)$ *such that* $r(T_i) > 1$, *for all* $i \geq 1$, *then there is a dense set* $D \subset X$ *such that for every* $n \geq 1$ *and every positive integers* $i_1 < i_2 < \ldots < i_n$ *the n-tuple orbit* $\mathrm{Orb}(\{T_{i_j}\}_{j=1}^{n}, x)$ *will tend to infinity for every* $x \in D$.

COMPETING INTERESTS

The authors declare that no competing interests exist.

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