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N-TUPLE ORBITS TENDING TO INFINITY FOR HILBERT SPACE OPERATORS

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Abstract. In [2] we gave some results on the existence of a dense set of vectors each having an n-tuple orbit tending to infinity for sequences of mutually commuting bounded linear operators acting on an infinite-dimensional complex Banach space. In this paper we will show that, in the case of operators on an infinite-dimensional complex Hilbert space, this type of set exists under weaker conditions.

1. Introduction

Let X be a complex Banach space and $B(X)$ the algebra of all bounded linear operators acting on X. For an operator $T \in B(X)$, $\sigma(T)$, $\sigma_{\mathrm{p}}(T)$, $\sigma_{\mathrm{ap}}(T)$ and $r(T)$ will denote the spectrum, the point spectrum, the approximate point spectrum and the spectral radius of the operator T, respectively.

If $T_1, T_2, \ldots, T_n \in B(X)$ are mutually commuting operators, the n-tuple orbit (or the orbit under the n-tuple $\mathbf{T} = (T_1, T_2, \ldots, T_n)$) of the vector $x \in X$ is the set

$$\mathrm{Orb}(\{T_i\}_{i=1}^n, x) = \mathrm{Orb}(\mathbf{T}, x) = \left\{T_1^{k_1} T_2^{k_2} \ldots T_n^{k_n} x : k_i \geq 0; 1 \leq i \leq n\right\}. \quad (1.1)$$

As in [2], we adopt the following definition given in [7]: the n-tuple orbit *tends to infinity* if

$$\lim_{k_i \to \infty} \left\| T_1^{k_1} T_2^{k_2} \ldots T_n^{k_n} x \right\| = \infty, \text{ for all } k_j \geq 0,\ j \neq i,\ 1 \leq i, j \leq n.$$

For $n = 1$, the n-tuple orbit (1.1) reduces to a simple sequence of form

$$\mathrm{Orb}(T, x) = \left\{T^k x : n = 0, 1, 2, \ldots\right\}.$$

This sequence is usually referred as *single orbit* (or simply *orbit*) of the vector $x \in X$ under the operator $T \in B(X)$.

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The existence of single orbits tending to infinity for operators on infinite dimensional spaces, along with the properties of the set of all vectors with this type of orbits under a given operator, in great extent is studied in [1], [5] and [6]. In some sense, the best possible result in this direction asserts that, for a power unbounded operator $T \in B(X)$ which satisfies

$$\sum_{k=1}^{\infty} \frac{1}{\|T^k\|} < \infty,$$

there is a dense set $D \subset X$ such that $\mathrm{Orb}(T, x)$ tends to infinity, for every $x \in D$ ([5, Corollary V.37.16]). In [3], the authors have shown that, given a sequence of operators $(T_i)_{i \geq 1}$ in $B(X)$ such that

$$\sum_{k=1}^{\infty} \frac{1}{\|T_i^k\|} < \infty, \text{ for all } i \geq 1,$$

then there is a dense set $D \subset X$ such that $\mathrm{Orb}(T_i, x)$ tends to infinity, for every $x \in D$ and $i \geq 1$ ([3, Corollary 10]). If, in addition, $(T_i)_{i \geq 1}$ is a sequence of mutually commuting operators in $B(X)$ with at least one of the properties (P.1) and (P.2) in Lemma 2 bellow, then the n-tuple orbit $\mathrm{Orb}(\{T_{i_j}\}_{j=1}^n, x)$ tends to infinity, for all $n \geq 2$, all positive integers $i_1 < i_2 < \ldots < i_n$ and all $x \in D$ ([2, Corollary 2.6]).

In this paper we will show that, for Hilbert space operators, the above conclusion holds under the assumption that

$$\sum_{k=1}^{\infty} \frac{1}{\|T_i^k\|^2} < \infty, \text{ for all } i \geq 1.$$

2. Preliminary Results

Throughout the rest of the paper, we assume that the spaces are complex and infinite dimensional.

Lemma 1. [5, Lemma V.37.15] *Let $\varepsilon > 0$ and $(a_i)_{i \geq 1}$ be a sequence of positive numbers satisfying $\sum_{i \geq 1} a_i < \varepsilon$. Then there is a sequence of positive numbers $(b_i)_{i \geq 1}$ such that $b_i \to \infty$ as $i \to \infty$ and $\sum_{i \geq 1} a_i b_i < \varepsilon$.*

Theorem 1. [5, Theorem V.37.17] *Let H and K be Hilbert spaces, $(T_i)_{i \geq 1}$ be a sequence of operators in $B(H, K)$, $(a_i)_{i \geq 1}$ be a sequence of positive numbers such that $\sum_{i \geq 1} a_i^2 < \infty$ and $\varepsilon > 0$. Then:*

- *(i) there exist $x \in H$ such that $\|x\| \leq \left(\sum_{i \geq 1} a_i^2\right)^{1/2} + \varepsilon$ and $\|T_i x\| \geq a_i \|T_i\|$, for all $i \geq 1$,*
- *(ii) there is a dense subset of vectors $x \in H$ such that $\|T_i x\| \geq a_i \|T_i\|$, for all i sufficiently large.*

Corollary 1.1. [5, Corollary V.37.18] *Let H be Hilbert space and $T \in B(H)$ is such that $\sum_{k=1}^{\infty} \left\|T^k\right\|^{-2} < \infty$. Then there is a dense set $D \subset X$ such that $\operatorname{Orb}(T, x)$ tends to infinity for every $x \in D$.*

Lemma 2. [2, Lemma 2.1] *Let X be a Banach space and let $T_1, T_2, \ldots T_m \in B(X)$, $m \geq 2$, be mutually commuting operators with at least one of the following properties:*

$(P.1)$ *the operator T_i is bounded bellow, for every i;*
$(P.2)$ $(T_i^k - T_j^k)_{k \geq 0}$ *is a norm bounded sequence, for every i and j.*

If $x \in X$ is such that $\operatorname{Orb}(T_i, x)$ tends to infinity for every $i \in \{1, 2, \ldots, m\}$, then the n-tuple orbit $\operatorname{Orb}(\{T_{i_j}\}_{j=1}^{n}, x)$ tends to infinity, for every $1 < n \leq m$ and every $1 \leq i_1 < i_2 < \ldots < i_n \leq m$.

3. Main Results

Let $F = \{1, 2, \ldots, m\}$ for some $m \in \mathbb{N}$ $(m \geq 2)$ or $F = \mathbb{N}$.

Theorem 2. *If H is Hilbert space, $\{T_i : i \in F\} \subset B(H)$ and $\{(a_{i,j})_{j \geq 1} : i \in F\}$ is a family of sequences of positive numbers such that $\sum_{i \in F, j \geq 1} a_{i,j}^2 < \infty$, then for every open ball B in H there exists $k_0 \in \mathbb{N}$ and $x \in B$ such that*

$$\left\|T_i^k x\right\| \geq a_{i,k} \left\|T_i^k\right\|, \quad \textit{for all } i \in F \textit{ and } k \geq k_0.$$

Proof. Let B be an open ball in H and, for $i \in F$ and $k \in \mathbb{N}$, let $T_{i,k} = T_i^k$.

If $F = \{1, 2, \ldots, m\}$ for some $m \in \mathbb{N}$ $(m \geq 2)$, let $f_m : F \times \mathbb{N} \to \mathbb{N}$ be the bijective mapping defined with

$$f_m(i, j) = i + m(j - 1), \text{ for all } (i, j) \in F \times \mathbb{N},$$

and $g_m : \mathbb{N} \to F \times \mathbb{N}$ be its inverse mapping. If $(\alpha_n)_{n \geq 1}$ is a sequence of positive numbers and $(S_n)_{n \geq 1}$ is a sequence of operators in $B(H)$ defined with

$$\alpha_n = a_{g_m(n)} \text{ and } S_n = T_{g_m(n)}, \text{ for all } n \in \mathbb{N},$$

then $\sum_{n \geq 1} \alpha_n^2 = \sum_{i \in F, j \geq 1} a_{i,j}^2 < \infty$ and hence, by Theorem 1 (ii), applied on $(\alpha_n)_{n \geq 1}$ and $(S_n)_{n \geq 1}$, there are $x \in B$ and $n_0 \in \mathbb{N}$ such that

$$\|S_n x\| \geq \alpha_n \|S_n\|, \text{ for all } n \geq n_0. \tag{3.1}$$

Since $g_m : \mathbb{N} \to F \times \mathbb{N}$ is bijective, there is a unique pair $(i_0, j_0) \in F \times \mathbb{N}$ such that $n_0 = i_0 + m(j_0 - 1)$. Clearly, $n_0 \leq m + m(j_0 - 1) = mj_0$ and, if $k_0 = j_0 + 1$, then for every $(i, k) \in F \times \mathbb{N}$ such that $k \geq k_0$ we have

$$f_m(i, k) = i + m(k - 1) > m(k - 1) \geq m(k_0 - 1) = mj_0 \geq n_0.$$

Hence, by (3.1) we have

$$\left\|T_i^k x\right\| = \|T_{i,k} x\| = \left\|S_{f_m(i,k)} x\right\| \geq \alpha_{f_m(i,k)} \left\|S_{f_m(i,k)}\right\| = a_{i,k} \|T_{i,k}\| = a_{i,k} \left\|T_i^k\right\|,$$

for all $i \in F$ and $k \geq k_0$.

Now let $F = \mathbb{N}$, $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be the bijective mapping defined with

$$f(i, j) = \frac{(i + j - 2)(i + j - 1)}{2} + j, \text{ for all } (i, j) \in \mathbb{N} \times \mathbb{N},$$

and $g:\mathbb{N}\to\mathbb{N}\times\mathbb{N}$ be its inverse mapping. If $(\alpha'_n)_{n\geq1}$ is a sequence of positive numbers and $(S'_n)_{n\geq1}$ is a sequence of operators in $B(H)$ defined with

$$\alpha'_n = a_{g(n)} \text{ and } S'_n = T_{g(n)}, \text{ for all } n\in\mathbb{N},$$

then $\sum_{n\geq1}{\alpha'_n}^2 = \sum_{i\in F, j\geq1} a_{i,j}^2 < \infty$ and hence, by Theorem 1 (ii), applied on $(\alpha'_n)_{n\geq1}$ and $(S'_n)_{n\geq1}$, there are $x\in B$ and $n'_0\in\mathbb{N}$ such that

$$\|S'_n x\| \geq \alpha'_n \|S'_n\|, \text{ for all } n\geq n'_0. \tag{3.2}$$

Since f and g are bijective, there is a unique pair $(i'_0, j'_0)\in\mathbb{N}\times\mathbb{N}$ such that

$$n'_0 = \frac{(i'_0+j'_0-2)(i'_0+j'_0-1)}{2} + j'_0.$$

Let $k_0 = i'_0 + j'_0$. Then for every $i\in\mathbb{N}$ and $k\geq k_0$

$$f(i,k) = \frac{(i+k-2)(i+k-1)}{2} + k \geq \frac{(i'_0+j'_0-2)(i'_0+j'_0-1)}{2} + j'_0 = n'_0.$$

Hence, by (3.2)

$$\left\|T_i^k x\right\| = \|T_{i,k}x\| = \left\|{S'}_{f(i,k)}x\right\| \geq {\alpha'}_{f(i,k)} \left\|{S'}_{f(i,k)}\right\| = a_{i,k}\|T_{i,k}\| = a_{i,k}\left\|T_i^k\right\|,$$

for all $i\in F$ and $k\geq k_0$. □

Corollary 2.1. *If H is Hilbert space and $\{T_i : i\in F\}\subset B(H)$ is a set of operators such that $\sum_{k=1}^{\infty}\left\|T_i^k\right\|^{-2}<\infty$, for all $i\in F$, then there is dense set $D\subset H$ such that $\mathrm{Orb}(T_i,x)$ tends to infinity for every $i\in F$ and every $x\in D$. If, in addition, the set $\{T_i : i\in F\}$ consists of mutually commuting operators that satisfy at least one of the conditions (P.1) and (P.2) in Lemma 2, then for every $n\in F$, $n\geq2$, every $i_1,i_2,\ldots,i_n\in F$ such that $i_1<i_2<\ldots<i_n$ and every $x\in D$, the n-tuple orbit $\mathrm{Orb}(\{T_{i_j}\}_{j=1}^n, x)$ tends to infinity.*

Proof. By Lemma 2 it is enough to show the first assertion. Let B be an open ball in H with radius $\varepsilon>0$. For $i\in F$, let $\varepsilon_i>0$ be such that

$$\varepsilon_i\left(\sum_{k=1}^{\infty}\frac{1}{\left\|T_i^k\right\|^2}\right) < \frac{\varepsilon}{2^{i+1}}.$$

By Lemma 1 there is a family of sequences of positive numbers $\{(b_{i,k})_{k\geq1} : i\in F\}$ such that $b_{i,k}\to\infty$ as $k\to\infty$ and

$$\sum_{k=1}^{\infty}\frac{\varepsilon_i b_{i,k}}{\left\|T_i^k\right\|^2} < \frac{\varepsilon}{2^{i+1}}, \text{ for all } i\in F. \tag{3.3}$$

For $(i,k)\in F\times\mathbb{N}$ put $a_{i,k} = \varepsilon_i^{1/2} b_{i,k}^{1/2}\left\|T_i^k\right\|^{-1}$. Then, by (3.3)

$$\sum\nolimits_{i\in F, k\geq1} a_{i,k}^2 = \sum_{i\in F}\sum_{k=1}^{\infty}\frac{\varepsilon_i b_{i,k}}{\left\|T_i^k\right\|^2} < \sum_{i\in F}\frac{\varepsilon}{2^{i+1}} < \frac{\varepsilon}{2},$$

and hence, by Theorem 2, there are vector $x\in B$ and $k_0\in\mathbb{N}$ such that

$$\left\|T_i^k x\right\| \geq a_{i,k}\left\|T_i^k\right\| = \varepsilon_i^{1/2} b_{i,k}^{1/2}\left\|T_i^k\right\|^{-1}\left\|T_i^k\right\| = \varepsilon_i^{1/2} b_{i,k}^{1/2},$$

for all $i \in F$ and $k \geq k_0$.

Letting $k \to \infty$, we obtain that $\left\|T_i^k x\right\| \to \infty$ as $k \to \infty$, for all $i \in F$. □

In [4] the authors have shown that, given a sequence of bounded linear operators $(T_i)_{i \geq 1}$ on a Hilbert space H, for which there is $\beta > 0$ such that

$$(\sigma_{\mathrm{ap}}(T_i) \backslash \sigma_{\mathrm{p}}(T_i)) \cap \{\lambda \in \mathbb{C} : |\lambda| > \beta + 1\} \neq \varnothing, \text{ for all } i \geq 1,$$

there is a dense set $D \subset H$ such that the single orbit $\mathrm{Orb}(T_i, x)$ tends to infinity for every $i \geq 1$ and every $x \in D$ ([4, Corollary 3.1]). If in addition, the sequence consists of mutually commuting operators having at least one of the properties (P.1) and (P.2) in Lemma 2, then the n-tuple orbit $\mathrm{Orb}(\{T_{i_j}\}_{j=1}^n, x)$ will tend to infinity for every integer $n \geq 2$, all integers $i_1, i_2, \ldots, i_n$ such that $1 \leq i_1 < i_2 < \ldots < i_n$ and every $x \in D$. On the other hand, Corollary 1.1 implies that their existence does not rely on the inner structure of the spectra (or some specific parts of the spectra) of the operators. More precisely, as in the case of Banach space operators, for Hilbert spaces operators we have the following two results which now can be derived as corollaries of Corollary 1.1.

Corollary 2.2. *If H is Hilbert space and $\{T_i : i \in F\} \subset B(H)$ is a set of operators such that $r(T_i) > 1$ for all $i \in F$, then there is dense set $D \subset H$ such that $\mathrm{Orb}(T_i, x)$ tends to infinity for every $i \in F$ and every $x \in D$. If, in addition, the set $\{T_i : i \in F\}$ consists of mutually commuting operators that satisfy at least one of the conditions (P.1) and (P.2) in Lemma 2, then for every $n \in F$, $n \geq 2$, every $i_1, i_2, \ldots, i_n \in F$ such that $i_1 < i_2 < \ldots < i_n$ and every $x \in D$, the n-tuple orbit $\mathrm{Orb}(\{T_{i_j}\}_{j=1}^n, x)$ tends to infinity.*

Proof. Again, by Lemma 2 it is enough to show the first assertion. Let $i \in F$. Since $r(T_i) > 1$ there is $\lambda_i \in \sigma(T_i)$ so that $|\lambda_i| > 1$. By the Spectral Mapping Theorem $\lambda_i^n \in \sigma(T_i^n)$, for every $n \in \mathbb{N}$. Hence, $|\lambda_i|^n \leq r(T_i^n) \leq \|T_i^n\|$. This would imply that $\sum_{n=1}^{\infty} \|T_i^n\|^{-2} \leq \sum_{n=1}^{\infty} |\lambda_i|^{-2n} < \infty$. Now the conclusion follows from Corollary 2.1. □

Corollary 2.3. *If H is Hilbert space and $\{T_i : i \in F\} \subset B(H)$ is a set of invertible and mutually commuting operators such that $r(T_i) > 1$, for all $i \in F$, then there is dense set $D \subset H$ such that the n-tuple orbit $\mathrm{Orb}(\{T_{i_j}\}_{j=1}^n, x)$ tends to infinity for every $n \in F$, every $i_1, i_2, \ldots, i_n \in F$ such that $i_1 < i_2 < \ldots < i_n$ and every $x \in D$.*

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